International Diploma in Business

Module DB105

Principles of Quantitative Methods

Student Notes

Modification History

Revision	Date	Revision Description
V0.2	May 2003	For issue and review.
V1.0	August 2003	For issue
2007	1 December 2006	Amended.

Trademarks

Module DB105

Principles of Quantitative Methods

Introduction to Module DB105

This module is made up of 12 topics.

The information contained in each topic is an overview and should be used in conjunction with the Assessment and Syllabus Guide and the required textbook for this module which is:

Swift, L. and Piff, S. (2005) *Quantitative Methods for Business, Management and Finance.* Palgrave Macmillan.

This textbook can be purchased from: www.palgrave.com

The companion website for this book has additional student and lecturer resources available at: www.palgrave.com/business/swift/

Additional textbooks:

Francis, A. (2004) *Business Maths and Statistics* Thomson Learning.

Rowe, R.N. (2002) *Refresher in Basic Mathematics* 3rd ed. Thomson Learning.

Learning Outcomes

	Specific Learning Outcomes (LOs)	**Demonstrable Knowledge**
LO 1	Knowledge and understanding of mathematical and statistical rules, concepts and techniques.	Knowledge and understanding.
LO 2	Ability to present and analyse typical business data.	Application and problem-solving. Analysis. Communication.

Indicative Content

1. Applying the four rules of numeracy to whole numbers, fractions and decimals.
2. Expressing numbers in standard form.
3. Multiplying and dividing negative numbers.
4. Apply calculations to:
   - compare numbers using ratios, proportions and percentages;
   - obtain values for simple financial transactions involving purchases, wages, taxation, and discounts;
   - calculate values using simple and compound interest;
   - convert foreign currency.
5. Perform calculations involving roots and powers.
6. Evaluate terms involving a sequence of operations and use of brackets.
7. Interpret and evaluate formulae.
8. Approximate data using rounding, significant figures.
9. Determine absolute and relative error.
10. Use algebraic methods to:
    - solve linear and simultaneous equations;
    - solve quadratic equations using factorisation and formulae;
    - determine the equation of a straight line through two points;
    - determine the gradient and intercept of a straight line.
11. Construct and use graphs applying general rules and principles of graphical construction including axes, choice of scale and zero.
12. Apply statistical methods to:
    - distinguish discrete and continuous variables;
    - recognise and use sigma notation for summation;
    - classify and present data in tabular and chart form;
    - determine and interpret mean, median, mode, quartile deviation and standard deviation for raw and grouped data;
    - present several sets of data on one graph for comparison;
    - construct statistical graphs for time series, determine a trend using moving averages and make a simple forecast;
    - measure uncertainty using probability techniques; distinguish between probability based on equally likely events or historical data and subjective probability, find outcomes and events using Venn diagrams, combine events using the operators AND, OR

and NOT, distinguish between mutually exclusive and independent events;

- present data in appropriate graphical form, with suitable headings, keys, labels and the source of data; recognise poor examples of graphical presentation.

Module Contents

Topic 1 Arithmetic and Ratios

1.1 Rules of Mathematical Operations

All mathematical expressions should be solved in the following order.

1. Groupings, i.e. brackets, innermost bracket first, moving outwards.
2. Multiplication and division, working from left to right.
3. Addition and subtraction, working from left to right.

1.1.1 Examples

1a. [(10 – 6) x 3] + 2
= [4 × 3] + 2
= 12 + 2
= 14

1b. {12 – [40 ÷ (7 – 2)]} × 3
= {12 – [40 ÷ 5]} × 3
= {12 – 8} × 3
= 4 × 3
= 12

2. When there are no groupings:
   a. If an expression has only addition and subtraction, work from left to right.

      Example: 58 + 12 – 4
      = 70 – 4
      = 66

   b. If an expression has only multiplication and division, work from left to right.

      Example: 15 ÷ 3 × 4
      = 5 × 4
      = 20

   c. If an expression has addition, subtraction, multiplication and division, work all multiplication and division FIRST, and then work from left to right.

      Example: 11 + 2 × 6 – 50 ÷ 5
      = 11 + 12 – 10
      = 23 – 10
      = 13

1.2 Fractions

The rule mentioned previously also applies here.

Examples:

a. $1\frac{2}{3}-\frac{1}{2}$

$= \frac{5}{3}-\frac{1}{2}$

$= \frac{10}{6}-\frac{3}{6}$

$= \frac{7}{6}$

$= 1\frac{1}{6}$

b. $1\frac{3}{4}-2\frac{1}{2}$

$= \frac{7}{4}-\frac{5}{2}$

$= \frac{7-10}{4}$

$= \frac{-3}{4}$

c. $(2\frac{1}{3}-1\frac{2}{6}) \div 1\frac{1}{4}$

$= (\frac{7}{3}-\frac{8}{6}) \div \frac{5}{4}$

$= (\frac{14}{6}-\frac{8}{6}) \div \frac{5}{4}$

$= \frac{6}{6} \div \frac{5}{4}$

$= 1 \times \frac{4}{5}$

$= \frac{4}{5}$

d. $(3\frac{1}{5}+1\frac{2}{3}) \times 3\frac{1}{2}$

$= (\frac{16}{5}+\frac{5}{3}) \times \frac{7}{2}$

$= (\frac{48}{15}+\frac{25}{15}) \times \frac{7}{2}$

$= \frac{73}{15} \times \frac{7}{2}$

$= \frac{511}{30}$

$= 17\frac{1}{30}$

1.2.1 Conversion of Decimal Numbers to a Fraction in the Lowest Terms

Examples:

a. $0.125 = \frac{125}{1000}$

$= \frac{1}{8}$

b. $0.20 = \frac{20}{100}$

$= \frac{1}{5}$

c. $0.314 = \frac{314}{1000}$

$= \frac{157}{500}$

1.2.2 Conversion of a Fraction to a Percentage

Examples:

a. $\frac{1}{4}$

$= \frac{1}{4} \times 100$

$= 25\%$

b. $\frac{2}{3}$

$= \frac{2}{3} \times 100$

$= 66.66\%$

1.3 Standard Form

When an answer is expressed in standard form it will be A x 10^n where:

$1 \le A < 10.$

Example:

1a. 2056 becomes 2.056 x 10^3
(when the decimal is moved to the left, n is positive)

1b. 0.0038 becomes 3.8 x 10^{-3}
(when the decimal is moved to the right, n is negative)

1.4 Significant Figure

The accuracy of an answer will depend on the number of significant figures. Rules determining significant digits:

1. Zeros at the end of a whole number that has no decimal point are not significant.
2. All non-zeros are significant, e.g. 589 (3 significant numbers).
3. Zeros in between non-zeros are significant, e.g. 609 (3 significant numbers).
4. Zeros in front of non-zeros are not significant, e.g. 0.0078 (has only 2 significant numbers).
5. Zeros after the decimal point preceded by non-zeros are significant e.g. 93.04 has 4 significant numbers).

Examples:

Write the following numbers according to the number of significant figures indicated in the brackets.

1. 89.308 (1) = 90
2. 56.8901 (3) = 56.9
3. 0.08745 (1) = 0.09
4. 0.003516 (4) = 0.003516

1.5 Exercise

Attempt all the questions in this exercise without using a calculator.

1. Calculate the following:

    a. 40 x 5 + 50 x 6 – 7 x 60
    b. (57 + 43 – 7 x 5) x 20 + 4 x 90
    c. (325 – 127) ÷ 9 + 136 – 11 x 11
    d. 16 ÷ [18 – (32 – 100 ÷ 5) ÷ 6]
    e. 1 ÷ 1 + 0 ÷ 26 + 26 ÷ 1
    f. 7 x 3 – 77 ÷ 11 + 16
    g. 56 ÷ 8 + (47 – 17) ÷ 5 – 13
    h. (32 x 5 – 60) x 25 – 90 x 15
    i. [4 x 15 + 72 ÷ 8 – (47 – 23) ÷ 6] x 2
    j. 105 ÷ 5 + 15 x 3 – (6 x 12 – 54 ÷ 9)
    k. 79 – 6 x 81 ÷ 9 + 65 ÷ 13 – 11
    l. (640 ÷ 80 + 54 ÷ 6) x 20 – 840 ÷ 7

2. Express the following numbers in standard form.

    a. 237
    b. 5,600
    c. 912,400
    d. 612,006
    e. 28,000,000
    f. 0.0435
    g. 0.00077
    h. 0.008306
    i. 0.296
    j. 74.8
    k. 159.4
    l. 70,600

3. Express the following numbers in ordinary notation.

    a. $6.37 \text{ x } 10^{3}$
    b. $4.213 \text{ x } 10^{-3}$
    c. $8.1 \text{ x } 10^{-5}$
    d. $1.729 \text{ x } 10^{4}$
    e. $3.82 \text{ x } 10^{-1}$
    f. $9.8 \text{ x } 10^{6}$
    g. $5.09 \text{ x } 10^{-3}$
    h. $2.47 \text{ x } 10^{2}$
    i. $3(4.7 \text{ x } 10^{-2})$
    k. $0.7\ (1.2 \text{ x } 10^{3})$

4. Calculate the following:

    a. $\frac{1}{3} \text{ x } \left(\frac{1}{4} - \frac{1}{12} + \frac{1}{2}\right)$

    b. $1\frac{3}{4} \text{ x } \left(\frac{4}{9} + \frac{2}{3}\right) \text{ x } \left(1\frac{1}{5} - \frac{1}{2}\right)$

    c. $\frac{2}{3} \text{ x } \frac{1}{4} - \frac{1}{12} \div \frac{1}{2}$

    d. $\frac{2}{3} \text{ x } \frac{1}{4} + \frac{5}{12} \text{ x } \frac{7}{15}$

    e. $\left(\frac{1}{5} + \frac{1}{3}\right) \div \frac{2}{3} \div \frac{2}{5}$

5. Express the following as fractions in their lowest terms.

a. 0.6 b. 0.75 c. 0.32 d. 0.36 e. 0.025 f. 1.5
g. 0.006 h. 0.105 i. 3.75 j. 0.0125 k. 15.25 l. 84.625

6. Convert the following fractions into decimals.

a. $\frac{9}{50}$ b. $\frac{1}{20}$ c. $\frac{7}{25}$ d. $\frac{3}{8}$ e. $\frac{11}{40}$ f. $\frac{13}{8}$
g. $\frac{67}{80}$ h. $\frac{215}{80}$ i. $\frac{19}{16}$ j. $\frac{245}{280}$ k. $\frac{19}{32}$ l. $2\frac{31}{50}$

7. Round off the following to i. the nearest whole number
ii. 1 decimal place
iii. 2 decimal places
iv. 3 decimal places

a. 5.42467 b. 15.82416 c. 7.86295 d. 130.82938

8. Express the following numbers correct to the number of significant figures indicated in the brackets.

a. 3.084 (2) b. 1.48356 (4) c. 0.00346 (1)
d. 6.047 (1) e. 3.14159 (2) f. 0.57438 (2)
g. 0.05678 (3) h. 217.006 (5) i. 3.598 (3)
j. 15.7037 (4) k. 0.00356 (2) l. 5.98 (2)

9. Evaluate the following giving your answer in standard form.

a. $(2 \text{ x } 10^{-16}) \text{ x } (3 \text{ x } 10^{-17)}$

b. $(3 \text{ x } 10^{-16}) / (4 \text{ x } 10^{-17})$

c. $(2 \text{ x } 10^{-16}) + (3 \text{ x } 10^{-17})$

d. $(2 \text{ x } 10^{-16}) - (3 \text{ x } 10^{-17})$

e. $(2 \text{ x } 10^{-16})^2$

f. $\sqrt[3]{8 \times 10^{27}}$

1.6 Ratio Proportion

Fractions or respective shares of a total are expressed in the form of x:y.

Example 1:

Partners Tan and Chan have agreed to share their net profit in the ratio of 2:3. If the net profit for the year ended December 1993 is $9000, what is their respective share?

Tan's share is $\frac{2}{5} \times 9{,}000$ = $3,600

Chan's share is $\frac{3}{5} \times 9{,}000$ = $5,400

Example 2:

XYZ Ltd is issuing bonus shares to existing shareholders. Bonus shares will be distributed as follows: For every four shares held, one share will be allocated. Ang and Pang are shareholders holding 10,000 and 25,000 shares respectively. How many bonus shares will each receive?

Ang will receive $\frac{10{,}000}{4} \times 1$ = 2,500 bonus shares

Pang will receive $\frac{25{,}000}{4} \times 1$ = 6,250 bonus shares

1.7 Exercise

1. In a shop, the ratio of the number of motorcycles to the number of scooters is 7:3. Given that there are 45 scooters in the shop, calculate the number of motorcycles.
2. A sum of money is divided in the ratio 2:3:7. If the largest share is $112, calculate the smallest share.
3. A sum of money was divided between Allan and Bill in the ratio 5:3.
    a. Express Bill's share as a % of the total sum.
    b. If Allan received $5 more than Bill, how much did Bill receive?

4. A sum of money is divided among three persons in the ratio 12:13:7. The smallest share is $91. Find the total sum and the largest share.

5. In an election, 85% of those entitled to vote, did so. There were three candidates and the votes for these candidates were divided in the ratio 5:3:1. Given that 22,950 people voted, calculate

   a. The number who voted for the candidate with the most votes.

   b. The number who were entitled to vote.

6. The ratio of money of student A to B is 3:4 and that of B to C is 5:6. Calculate the ratio of A to C.

7. A bonus of $1000 is to be divided among four employees, A, B, C, D. Employees C and B get twice and thrice the amount A gets, respectively. D gets twice as much as C gets. How much do they each get?

Topic 2 – Algebra

2.1 Linear Equations

Examples

1. $5x - 3 = 2x + 6$
   $5x - 2x = 6 + 3$
   $3x = 9$
   $x = \frac{9}{3}$
   $x = 3$

2. $\frac{1}{3}x + 4 = 15$
   $\frac{1}{3}x + \frac{12}{3} = 15$
   $\frac{(x+12)}{3} = 15$
   $x + 12 = 45$
   $x = 45 - 12$
   $x = 33$

3. $5(x + 1) - 2(x - 1) = 10$
   $5x + 5 - 2x + 2 = 10$
   $3x + 7 = 10$
   $3x = 10 - 7$
   $3x = 3$
   $x = \frac{3}{3}$
   $x = 1$

4. If $p = \frac{2q}{3}$
   a. Find p when $q = 6$

   $$p = \frac{2q}{3} = \frac{2(6)}{3} = 4$$

   b. Find q when $p = 1$

   $$p = \frac{2q}{3}$$
   $$1 = \frac{2q}{3}, \qquad q = \frac{3}{2} \quad \text{or} \quad 1\frac{1}{2}$$

2.2 Exercise

1. Solve the following linear equations.

   a. $6(3x - 5) = \frac{12}{5}$

   b. $2(x - 1) - 3(x - 5) = 6(5x + 7)$

   c. $1 - 2x = x - 8$

   d. $1 - 2x = 3x - 9$

   e. $2(3 - x) = 5(2x - 3)$

   f. $2(x - 5) - 5(x - 3) = 0$

   g. $\frac{x}{2} + 2 = \frac{2x}{3}$

   h. $\frac{(x+3)}{4} = \frac{(2x-3)}{5}$

2. Given that $y = \frac{a}{(a+x)}$, find y when $a = \frac{1}{4}$, $x = \frac{5}{16}$

3. Given that $(2x - y)(m + 5) = m(x - 1)$ and $x = 3$, when $y = -1$, find m.

4. Given that $y = k\sqrt{x}$ and $y = 5$ when $x = 9$, calculate

   a. y when $x = 36$

   b. x when $y = 15$

5. Given that $s = 32t^2$, calculate

   a. s when $t = \frac{3}{4}$

   b. t when $s = 2$

2.3 Straight Lines

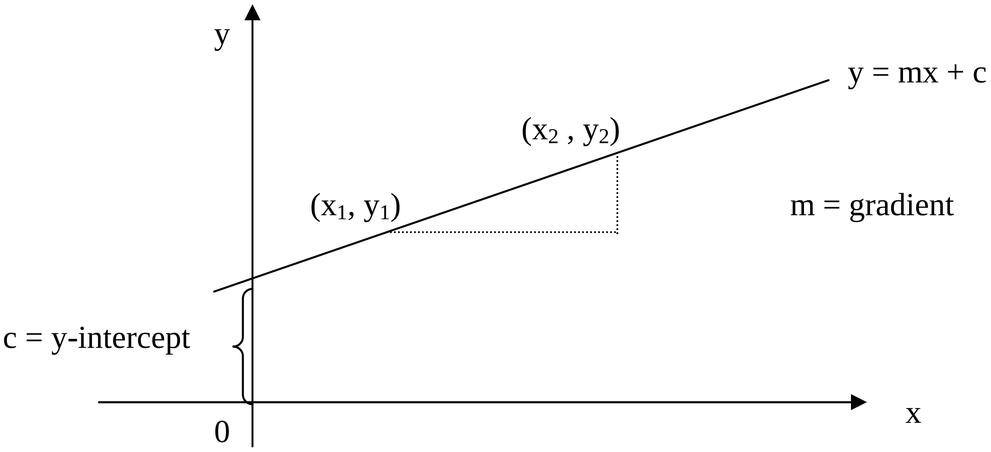
The equation of a straight line is a linear function of the form:

$$y = mx + c$$

where:

m = gradient or slope of the straight line given by m = $\frac{y_2 - y_1}{x_2 - x_1}$

c = y-intercept



2.3.1 Finding the Linear Equation Joining through Two Points

Example 1:

Given $(x_1, y_1) = (5, 3)$ and $(x_2, y_2) = (1, 6)$ determine the linear equation.

Solution:

- *Step 1* – find the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-3}{1-5}$$

$$= \frac{3}{-4} = -\frac{3}{4}$$

- *Step 2* – find c by substituting any of the co-ordinates into the equation

  y = mx + c.

$$3 = -\frac{3}{4(5)} + c$$

$$= -\frac{15}{4} + c$$

$$3 + \frac{15}{4} = c$$

$$c = \frac{12}{4} + \frac{15}{4} = \frac{27}{4}$$

We get $y = -\frac{3}{4}x + \frac{27}{4}$

<u>or</u> $4y = -3x + 27$

Example 2:

Find the equation of the straight line with y-intercept 3 and parallel to the straight line y = 2x + 5.

Solution:

c = 3, given
m = 2, since two parallel lines have the same gradient

So, the equation is y = 2x + 3.

2.4 Exercise

1. Find the equation of the straight line passing through (2, –2) and (7, 8).
2. Find the equation of the straight line through (3, 5) with gradient –4.
3. A straight line whose equation is y = –2x + 12, cuts the x-axis at A and the y-axis at B. Find the co-ordinates of A and B.
4. The gradient of the line joining (6, k) and (4, 1) is $\frac{3}{5}$. Find k.
5. A straight line passes through (1, 5) and has a gradient of 3. Find its equation. If it also passes through (h, 10 + h), find h.

6. The gradient of the line joining (7, k) and (k, 3) is $\frac{4}{5}$. Find k.
7. The equation of a straight line is $3y + 2x + 6 = 0$. Calculate:
    a. the gradient.
    b. the co-ordinates of the point where it cuts the x-axis.
    c. the co-ordinates of the point where it cuts the line $y = 4$.
    d. the equation of the line parallel to it and passing through (5, 2).

2.5 Simultaneous Equations

They are a set of equations which have more than one value to be found.

Solving at least two equations is effectively finding their meeting point.

Example 1:

$y = x + 4$
$y = 3x – 2$

Let $y = x + 4$ i (multiply by –1 to cancel off by)
$y = 3x – 2$ ii

$–y = –x – 4$ i
$y = 3x – 2$ ii

$0 = 2x – 6$
$6 = 2x$
$x = 3$

Substitute $x = 3$ into equation (i)

$y = 3 + 4$
$y = 7$.

Answer (3, 7).

2.6 Exercise

1. Solve the following simultaneous equations.
   a. $3x - 4y = 10$, $5x + 7y = 3$
   b. $5x + 3y = 3$, $3x + 2y = -1$
   c. $2x + 5y = 25$, $3x - 2y = 9$
   d. $3x + 3.2y = 40$, $2x - 3.2y = 0$
   e. $3x + 5y = 11$, $2x - 3y = 20$
   f. $3x - 2y = 13$, $2x + 3y = 0$
   g. $3x + 2y = -4$, $x - 3y = 17$
   h. $3x + 4y = 22$, $2x + 6y = 23$
   i. $3x - 4y = 25$, $4x - 5y = 32$
   j. $3x + 2y = 0$, $2x - 3y = 26$
   k. $3x + 2y = 7$, $x - y = 1.5$
   l. $4x + 5y = -1$, $3x + 2y = 8$
   m. $3x + 4y = 5$, $2x - 3y = 9$
   n. $4x - 7y = 23$, $6x + 2y = -3$
   o. $3x + 2y = 1$, $4x - y = 16$

2. John spends \$8.40 in buying 100 postage stamps. If x of them are 7 cent stamps and the remaining y are 9 cent stamps, find how many of each type of stamp he bought.

3. A boy buys 5 bars of chocolate at x cents each and y bags of toffee at 20 cents each. The total cost is \$4. Find x and y given that $x + y = 35$.

4. A woman buys 20 lemons at x cents each and 30 oranges at y cents each. She packs them into bags which contain 4 lemons and 6 oranges and sells the bags at $(5x + 8y)$ cents each. Find x, y if the amount of money she spent on the fruit is \$1.90 and the money she made by selling all the fruit is \$2.50.

2.7 Quadratic Equations

It assumes the function of $ax^2 + bx + c = 0$. To solve x we can use either:

1. Factorisation method.
2. Formula method.
3. Graphical method.

2.7.1 Factorisation Method

Example:

a.
$x^2 + 3x - 4 = 0$
$(x + 4)(x - 1) = 0$
$x = -4, 1$

b.
$2x^2 - 7x - 4 = 0$
$(2x + 1)(x - 4) = 0$
$x = 4, -\frac{1}{2}$

c.
$x^2 - x = 0$
$x(x - 1) = 0$
$x = 0$ or $x - 1 = 0$
$x = 0$ and $x = 1$

2.7.2 Formula Method

When the quadratic equation cannot be factorised easily the formula method can be used.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1:

Before the formula can be used, ensure that the equation is in the order of $ax^2 + bx + c = 0$.

Using the example in b of Section 2.7.1 above, $2x^2 - 7x - 4$, we get:

$a = 2$
$b = -7$
$c = -4$

Substitute these values into the formula, we get:

$$\frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-4)}}{2(2)}$$

$$= \frac{7 \pm \sqrt{49 + 32}}{4}$$

$$= \frac{7 \pm \sqrt{81}}{4}$$

$$= \frac{7+9}{4} \quad \text{or} \quad \frac{7-9}{4}$$

$$= \frac{16}{4} \quad \text{or} \quad \frac{-2}{4}$$

$$= 4 \quad \text{or} \quad -\frac{1}{2}$$

Example 2:

$6x^2 + 8x - 6 = 0$ (give answer correct to 2 decimal places).

From the equation we get:

a = 6
b = 8
c = –6

Substituting in the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we get:

$$\frac{-8 \pm \sqrt{(8)^2 - 4(6)(-6)}}{2(6)}$$

$$= \frac{-8 \pm \sqrt{64+144}}{12}$$

$$= \frac{-8 \pm \sqrt{208}}{12}$$

$$= \frac{-8 \pm 14.4222}{12}$$

$$= \frac{-8 + 14.4222}{12} \quad \text{or} \quad \frac{-8 - 14.4222}{12}$$

$$= 0.53518 \quad \text{or} \quad -1.86851$$

$$= 0.54 \quad \text{or} \quad -1.87$$

2.7.3 Graphical Method

To get an accurate answer from a graph, we will need to draw as many points as possible. Quadratic function graphs will either assume minimum or maximum position.

$$x^2 + 3x - 4 = 0$$

Draw a graph to obtain the values of x when y = 0.

- *Step 1* – find the respective value of y for a range of values of x. Let the range be –4 to 2.

X	–4	–3	–2	–1	0	1	2
Y	0	–4	–6	–6	–4	0	6

- *Step 2* – from the graph, the curve will cut the x-axis at:

  $x = -4$ and $x = 1$.

2.8 Exercise

Solve the following quadratic equations.

1. $x^2 + 5x + 6 = 0$
2. $x^2 + 3x - 4 = 0$
3. $x^2 + 3x + 2 = 0$
4. $x^2 + x - 2 = 0$
5. $x^2 + 6x + 8 = 0$
6. $x^2 - 2x - 8 = 0$
7. $x^2 + 7x + 10 = 0$
8. $x^2 - 7x + 10 = 0$
9. $x^2 + 8x + 12 = 0$
10. $x^2 - x - 6 = 0$
11. $x^2 - x - 12 = 0$
12. $x^2 - 7x + 12 = 0$
13. $x^2 - 4 = 0$
14. $16x^2 - 1 = 0$
15. $9 - 25x^2 = 0$
16. $(x - 3)^2 = 4$
17. $2x^2 - 7x - 4 = 0$
18. $2x^2 + 5x - 3 = 0$
19. $6x^2 - x - 12 = 0$

Topic 3 – Simple Interest and Compound Interest

3.1 Simple Interest

3.1.1 Introduction

When money is borrowed or deposited in a bank, interest will have to be paid or received. Interest paid or received is calculated as a percentage called the 'rate' and the amount borrowed or deposited is called the 'principal'. Simple interest is interest earned only on the principal. There is no interest on interest.

3.1.2 Formula

Interest = Principal x Rate of interest x Time period

I = P x R x T

Examples:

1. Find the interest when $400 is invested for 9 months at 6% per annum.

$$\text{Interest} = 400 \times 0.06 \times \frac{9}{12}$$
$$= \$18$$

2. Find the principal if the simple interest on an amount, invested at 10% per annum for 4 years, is $240.

$$I = P \times R \times T$$
$$\$240 = P \times 0.10 \times 4$$
$$P = \frac{240}{0.4}$$
$$= \$600$$

3. How long does it take for $300 to amount to $450 at 5% per annum simple interest?

$$I = P \times R \times T$$
$$= 300 \times 0.05 \times T$$
$$T = \frac{150}{15}$$
$$= 10 \text{ years}$$

4. Mr Tan received $850 at the end of five years, when an amount was invested at 6% simple interest. What is the amount?

$$P + (P \times 0.06 \times 5) = \$850$$
$$P + 0.3P = 850$$
$$1.3P = 850$$
$$P = \$653.85$$

3.2 Compound Interest

3.2.1 Introduction

Compound interest refers to the return on the principal invested, assuming that at the end of the first time period, interest is added to the principal sum and the total amount then continues to earn interest in all subsequent periods. (Each time interest is credited it becomes part of the principal sum.) In short, interest earns interest.

3.2.2 The Future Value of a Current Sum

In lending or investing we part with a current amount in the hope of receiving a larger sum in the future. To find the future sum, we convert Present Value (PV) to the Future Value (FV), which is larger.

3.2.3 Formula

PV to FV
$S = P(1 + r)^n$

Where: P is present value/sum;
S is future value/sum;
r is compound interest rate;
n is time period.

Example:

1. When \$6,000 is invested at 5% per annum, what is the future sum receivable after four years?

$$\begin{aligned} S &= P\,(1 + r)^n \\ &= 6{,}000\,(1 + 0.05)^4 \\ &= \$7{,}293.04 \end{aligned}$$

2. Mr Tan invests \$3,000 of his gratuity received on retirement today, and a further \$2,000 at the end of the second year. If the interest rate is 7% per annum, what is his investment worth at the end of five years?

$$S = P(1 + r)^{n}$$
$$S = \$3{,}000(1 + 0.07)^5 = \$4{,}207.65 \quad \text{and} \quad S = \$2{,}000(1 + 0.07)^3 = \$2{,}450.09$$

$$\begin{aligned} \text{Total investment worth} &= \$4{,}207.65 + \$2{,}450.09 \\ &= \$6{,}657.74 \end{aligned}$$

3. How much must Mr Tong invest now at 3% compound interest if, after six years, the investment is to reach the sum of $10,000?

$$S = P(1 + r)^{n}$$

$$\$10{,}000 = P(1 + 0.03)^{6}$$

$$P = \frac{10000}{(1+0.03)^{6}}$$

$$= \$8{,}374.84$$

$8,374.84 must be invested now to reach a future sum of $10,000

4. What is the prevailing interest rate if $5,000 grows to $7,000 in four years?

$$S = P(1 + r)^{n}$$

$$7000 = 5000(1 + r)^{4}$$
$$(1 + r)^{4} = 7000 \div 5000$$
$$= 1.4$$
$$1 + r = \sqrt[4]{1.4}$$
$$= 1.08775$$
$$r = 0.08775$$
$$= 8.775\%$$

3.3 Discounting (Present Value of Dollars)

3.3.1 Introduction

A dollar in the future will be worth less than at present. To compare amounts of money received at different times, it is necessary to reduce them to the present-day equivalent. This process is called discounting cash flow. Discounting is the reverse process of compounding. It is changing a future value to a present value.

3.3.2 Formula

FV to PV

$$P = \frac{S}{(1+r)^{n}}$$

Where P is the present value/sum
S is the future value/sum
R is the interest rate
n is the time period

Example 1:

Mr Chan will receive $100 at the end of year 2. What is the $100 worth now, assuming the compound interest rate is 10%?

$$P = \frac{\$100}{(1+0.10)^2}$$

$$= \frac{100}{1.21}$$

$$= \$82.64$$

Note that discounting or converting a future sum to its present value will result in a smaller amount.

Example 2:

ABC Ltd bought office equipment. The terms of payment are $10,000 now, together with an instalment of $3,000 at the end of each of the next 2 years. The compound interest rate is 20% per annum. What would be the equivalent total cash payment today for this equipment?

$$P = \frac{\$3000}{(1+0.20)^1} \qquad P = \frac{\$3000}{(1+0.20)^2}$$

$$P = \$2{,}500 \qquad P = \$2083.33$$

The equivalent total cash payment today for the equipment is:

$10,000 + 2,500 + 2,083.33 = $14,583.33.

In other words, ABC Ltd could pay $14,583.33 today instead of $16,000 by instalments.

(Note that the $10,000 which is referred to as payable now is included in the present value.)

3.4 Exercise

1. Calculate the simple interest on $1,920 invested for 8 months at 5% p.a.
2. A bank charges $28 simple interest on a sum of money borrowed for 4 months. Calculate the sum of money if the rate of interest is 15% p.a.
3. A sum of $350 is invested at a rate of simple interest and amounts to $476 after 3 years. Calculate:
   a. the total interest;
   b. the rate % p.a.

4. Find the simple interest obtained when \$125 is invested at 8% p.a. for a period of 9 months.
5. When \$5,000 is invested at 4% p.a., what is the future sum to be received after 4 years?
6. Mr Ang invests \$2,000 at 6% p.a. He wants to invest \$3,000 more after 2 years at the same rate. How much would he receive after 6 years?
7. What should the rate of interest be if Mr. Lee needs \$5,000 after 3 years and he now has \$3,000?
8. I need \$20,000 4 years from now. How much money should I invest if the rate of interest is 5%?
9. A company will have to spend \$300,000 on a new plant in 2 years time. What single sum should be invested if the compounding is 6 monthly, and the rate of interest is 10% per annum?
10. A company borrowed \$6,000 from a bank at 13% p.a., compounded quarterly. How much is owed after 2 years?
11. Ms Tan will receive \$500 at the end of 5 years. What is her money worth now if the interest rate is 10% p.a.?
12. I want to invest some money in order to receive \$5,000 after 4 years, at 5% interest p.a. How much money do I need now?
13. A manufacturing company bought some machinery on loan. They need to pay \$7,000 now and \$2,000 every year for the next 3 years at 15% p.a. What is the price of the machinery now?
14. A sum of \$200,000 is borrowed from a bank. The interest rate is 5% for each half year.
    a. How much must be repaid, to the nearest \$100, if compound interest is applied for 5 years?
    b. If the interest rate was 12% p.a., compounded semi-annually, how much extra interest (to the nearest \$100) would have to be paid on the loan?
15. \$7,000 is invested at simple interest and it amounts to \$8,960 after 4 years. Calculate the rate per annum.
16. \$7,000 is invested, earning 1.5% per 2 months compound interest. Calculate the final value of the investment after 2 years.
17. \$500 is deposited in a bank and earns compound interest. After 2 years, it amounts to \$576.11. What is the interest rate p.a. correct to 2 decimal places?
18. A sum of \$55,000 earns \$5,000 interest after 6 months. What is the % rate p.a.?

19. a. A sum of $10,000 is invested in a bank at the compound interest rate of 8% p.a. Find the total sum received after 8 years.
    b. After the 8th year, the interest rate is changed. What interest rate would apply if the sum received after a further 2 years is $25,000?
20. A sum of $450 earns $19.50 interest after 8 months. What is the rate of interest per annum?
21. A sum of $5,000 is borrowed from a bank. The rate of interest is 1.5% per quarter (3 months). How much must be repaid after 2 years?
22. An amount is deposited in a bank earning compound interest. After 10 years, the amount is doubled. What is the rate of interest per annum?

Topic 4 – Profit and Loss

4.1 Profit and Loss

Profit is made when sales > cost or sales – cost = profit.

Loss is made when sales < cost.

Profit can be expressed in terms of cost (mark-up) or in terms of sales (margin).

4.1.1 Formula

$$\text{Mark-up} = \frac{\text{Profit}}{\text{Cost}} \times 100 \qquad \text{Margin} = \frac{\text{Profit}}{\text{Sales}} \times 100$$

Example 1:

Chong sells his car for $5,000 making a profit of 20% on cost. What is his profit?

Solution :

$$\text{Profit} = \frac{20 \times 5{,}000}{120}$$

$$= \$833.33$$

Example 2:

Lee Ltd reveals that their sales margin is 55% when goods are sold for $10,000. What is the cost of the goods?

Solution:

$$\text{Cost} = \frac{45 \times 10{,}000}{100}$$

$$= \quad \$4{,}500$$

4.2 Exercise

1. A dealer made a profit of 20% by selling an article for $630. Calculate the price he paid for it.
2. An agent bought a car for $4,000 and sold it for $4,500. Calculate his:
   a. mark-up;
   b. margin.

3. I sold an article for $300 making a profit of 10% on cost. What is my profit?
4. A man bought a picture for $325 and sold it at a profit of 12% on cost. What is the selling price?
5. An article is sold for $15.40, thereby making a profit of 12% on cost. Calculate the profit.
6. If a company's sales margin is 40% when sales are $25,000, what is the cost?
7. In a sale, all prices are reduced by 30%. Calculate the original price of an article whose sale price is $32.20.
8. A shopkeeper bought a TV for $560. A 30% profit was computed to establish the marked price. A 10% discount was allowed on the marked price for a cash sale. Calculate how much a customer had to pay for the TV set when paying cash.
9. The list price of a microwave oven is $1,200, while the cash price is $1,020. Calculate the % discount if a customer buys the oven for the cash price.
10. A shopkeeper sold a pair of shoes for $176. He made a profit of 10%. What price must he sell the next pair at in order to make a total profit of 15%?
11. In a sale, all prices are reduced by 40%. Calculate the original price of an article whose sale price is $48.
12. A lady buys an article at 10% discount, whose original price is $40. In addition she has to pay 15% tax. Calculate how much she has to pay in total.

4.3 Conversion of Foreign Currencies

Currencies are expressed in terms of one unit of currency to another.

Example 1:

1 RM = S$0.60, how much will 100 RM be exchanged for?

x = 100 RM
1 RM = S$0.60
x = 100×0.60
x = S$60

100 RM will be exchanged for S$60.

Example 2:

Assuming the same exchange rate, how much RM will be received in exchange for S$100?

1RM = S$0.60
x = S$100
0.60 = 100
x = S$166.67

Example 3:

The table gives the exchange rate for S$1.

Romania	102.5	leu
Italy	0.78	euro
Myanmar	50.3	kyat
Norway	5.67	krone

How much currency would be received in each of the following transactions:

1. S$2000 converted to Romanian currency

   x = S$2,000
   S$1 = 102.5 leu
   x = 102.5 x 2,000
   = 205,000 leu

2. 1000 euro converted to Krone

   x = €1,000
   S$1 = €0.78
   x = S$780

   1S$ = 5.67 krone
   x = 5.67 krone x 780
   = 4422.6 krone

4.4 Exercise

1. Below is the exchange rate of S$1 to the given currencies.

   US $: 0.71 £: 0.46 Aust $: 0.92 Can $: 0.96 NZ $: 1.02

   How much money would be received in the following transactions?
   a. S$500 converted to Aust. $
   b. US$210 converted to Nz $
   c. £350 converted to S$
   d. $52.04 converted to US$ if 3% commission is to be paid.

2. A bank exchanges French currency for English currency at the rate of €1.49 to the £.
    a. Calculate the amount in $ received in €, in exchange for £12.50.
    b. Calculate the amount received in exchange for €80.

3. A bank exchanges US currency for British currency at the rate of $1.60 to the £.
    a. Calculate the amount in $ received for £150.
    b. Calculate in £, the amount paid for $800 by a customer who also had to pay 2% commission for the transaction.

4. Recently, the exchange rates for money in two South American countries were:

    Argentina: 18.50 Australs = $1,   Paraguay: 401 Guaranies = $1

    a. How many Guaranies were equivalent to 1 Austral?
    b. In Argentina, 250 g of cheese was 3000 Guaranies per Kg. Find the cost of 100 g of cheese in Australs.
    c. In Paraguay, the cost of the same cheese was 3,000 Guaranies per Kg. Find the cost of 100 g of cheese in (i) Guaranies, (ii) Australs.

5. Below are the bank exchange rates for £1.

    France, Germany: €1.49,   USA: $1.46,   Japan: 159 Yen.

    How much currency would be obtained in each of the following transactions with 3% commission?

    a. €350 to Yen.
    b. €25 to $.

4.5 Depreciation

Depreciation means assets with finite lives lose value over time.

Factors such as wear and tear and obsolescence cause fixed assets to have a limited useful life.

Depreciation is therefore an expense.

The two most common methods of depreciation are:

- straight line;
- reducing balance.

Illustration 1a – Straight line method:

A car is bought for $10,000 and depreciates by 10% of the cost per annum. What is the depreciation expense over the years?

	$
Car cost	10,000
Depreciation expense for end year 1 (10% x 10,000)	1,000
Net book value for end year 1	9,000
Depreciation expense for end year 2 (10% x 10,000)	1,000
Net book value for end year 2	8,000
Depreciation expense for end year 3 (10% x 10,000)	1,000
Net book value for end year 3	7,000

Note that 1. Depreciation expense is constant and is $1,000 every year.

2. 10% per annum on cost will result in a constant depreciation expense as the cost is always $10,000.

Illustration 1b – Straight line method:

A car cost $20,000 and is estimated to have a useful life of 10 years, thereby enabling the owner to scrap it for $5,000. The depreciation expense can be calculated using the formula:

$$\frac{\text{Cost – scrap value}}{\text{Useful life}} = \frac{\$20{,}000 - \$5{,}000}{10}$$

$$= \$1{,}500 \text{ per annum}$$

Illustration 2 – Reducing balance method

A piece of machinery cost $15,000 and the depreciation policy is to take 10% of its value per annum, at the beginning of every year. What is the depreciation expense and respective book value for the next 3 years?

	$
Machinery	15,000
Depreciation expense for end year 1 (10% x 15 000)	1,500
Net book value for end year 1	13,500
Depreciation expense for end year 2 (10% x 13 500)	1,350
Net book value for end year 2	12,150
Depreciation expense for end year 3 (10% x 12 150)	1,215
Net book value for end year 3	10,935

Note that: 1. The depreciation expense is reducing.

2. The 10% is calculated on the *value* of assets which is not constant in the amount it reduces.

Illustration 3 – Alternative method:

$D = B(1 - r)^n$

where D = depreciated value
B = book value
r = % rate of depreciation
n = number of years

Using the previous example:

$$\begin{aligned} D &= 15{,}000\ (1 - 0.10)^3 \\ &= 15{,}000\ (0.90)^3 \\ &= \$10{,}935 \end{aligned}$$

4.6 Exercise

1. A fridge was bought for $1,500. It depreciates 5% on cost. For how much can it be sold after eight years?
2. I bought a TV for $2,000. After three years I sold it for $1,340. How much is the % depreciation on cost?
3. A company bought a computer for $3,500. It depreciated by 10%. What is its price after two years if the depreciation is:
   a. on cost;
   b. on its price every year?
4. A mainframe computer whose cost is $220,000 will depreciate to a scrap value of $12,000 in five years.

   What is the book value of the computer at the end of the 3rd year (to the nearest whole number) assuming a reducing balance method?
5. A special council amenity costs $500,000 and its useful life is planned over 25 years. If the reducing balance depreciation method is used at 10% rate, what is the book value of the amenity at the end of its life?

Topic 5 – Graph Drawing and Break-even Analysis

5.1 Exercise

1. Draw the graph of $y = x^2 - 5x + 5$ for $0 \leq x \leq 5$ at intervals of one unit. Use this graph to estimate the solution of $x^2 - 5x + 5 = 0$. Where does this graph meet $y = 5$?

2. Find the meeting point of the straight lines $3x - y = 2$ and $x - 2y + 8 = 0$ using the graph for $0 \leq x \leq 7$ at intervals of one unit.

3. Plot the following functions on the same graph using $-2 \leq x \leq 5$ at intervals of one unit; $y = 4x - 9$ and $y = 3 + 13x - 4x^2$. Find their meeting point.

4. Find the solutions of $3x^2 - 8x - 5 = 0$ using the graph for the range
   $-2 \leq x \leq 5$ at intervals of one unit.

5. Solve graphically $x + y = 4$, $3x - y = 8$.

5.2 Cost, Revenue and Profit Function

5.2.1 Cost Behaviour

For our analysis, cost can be classified into:

a) variable cost;
b) fixed cost;

giving us the equation:

Total cost = total variable cost + total fixed cost.

5.2.2 Variable Cost

Variable cost is a cost that changes when units produced change: variable cost per unit is uniform or constant, but total variable cost changes in direct proportion to the units produced.

Illustration – Behaviour of variable cost:

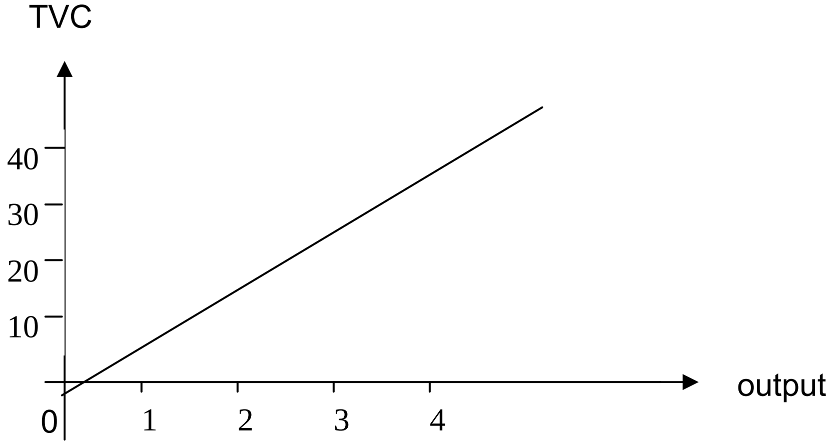
ABC Ltd employs operators at the rate of $10 per unit of output produced. Wage rate is an example of variable cost.

Output	Variable cost per unit	Total variable cost/wages
0	0	0
1	\$10 (\$10 ÷ 1)	\$10
2	\$10 (\$20 ÷ 2)	\$20
3	\$10 (\$30 ÷ 3)	\$30
-		-
-		-
-		-
-		-
100	\$10 (\$1,000 ÷ 10)	\$1,000

↓ Variable cost per unit is uniform

↓ Total variable cost varies in direct proportion to output

Examples of variable costs are raw material, commission, operators' wages. When the total variable cost is drawn on a graph we obtain:



5.2.3 Fixed Cost

Fixed costs behave in the opposite way to variable cost. Total fixed cost remains unchanged within a certain range, but fixed cost per unit reduces as units produced increases.

Illustration – Behaviour of fixed cost:

ABC Ltd rents its factory for $5,000 per annum.

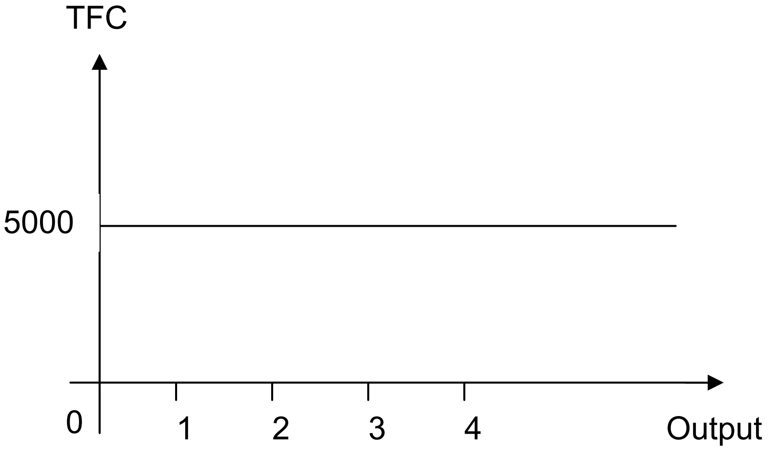
Output	Fixed cost per unit	Total fixed cost
0	-	$5,000
1	$5,000	$5,000
2	$2,500 (5,000 ÷ 2)	$5,000
3	$166.67 (5,000 ÷3)	$5,000
-		-
-		-
-		-
-		-
1,000	$50 ($5,000 ÷ 100)	$5,000

↓ Fixed cost gets smaller as output increases

↓ Total fixed cost remains fixed regardless of output

Examples of fixed costs are, supervisor's salary, building insurance.

When total fixed cost is plotted on the graph with its output we obtain:



Putting the two graphs together we can see:

Total cost	=	total variable cost + total fixed cost
TC	=	TVC + TFC

For our illustration, TC = \$5,000 + \$10q where q is units of output.

5.2.4 Revenue

Revenue or sales is a function of quantity sold and selling price per unit.

Revenue function → R = pq where R is revenue in total
p is selling price per unit
q is quantity sold

Profit function → R – TC = P where R is revenue
TC is total cost
P is profit

5.2.5 Break-even Analysis

'Break-even' means a situation of no profit or losses. In other words, revenue is equal to total cost. It is important to know the break-even output in businesses so that an output target can be made above break-even level, in order to ensure a profit.

Producing and selling above break-even means profit is being made.

Producing and selling below break-even means a loss is being made.

At break-even $\boxed{R = TC}$ or $\boxed{\text{Profit} = 0}$

There are two methods to determine the break-even point:.

a. Equation method.

b Graphical method.

a. Equation method

Illustration –break-even analysis:

PQR Ltd produces the sling bag model XXY. The variable cost per bag is \$20 and the fixed cost is \$8,000. How many bags must be sold in order to break even, assuming a selling price of \$45 per bag? What is the level of sales at break-even?

Revenue function = p.q

= $45q

TVC function = $20q

TC = $(20q + 8000)

R – TC = 0

$45q – $20q – $8,000 = 0

25q = 8,000

q = 320

PQR Ltd must sell 320 bags in order to break even.

The level of sales at break-even is:

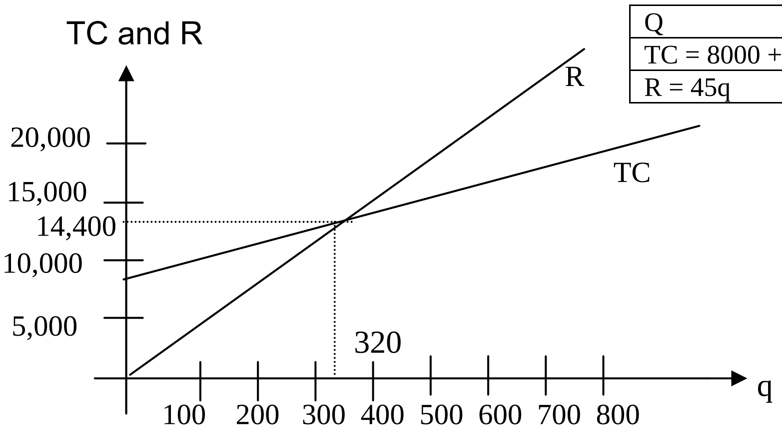
R = 45 x 320

= $14,400

b. Graphical method

Revenue and cost function are assumed to be linear in PQR Ltd's illustration. When the equations are linear we only need a maximum and a minimum point to plot the graph.

Q	0	200	400
TC = 8000 + 20q	0	9,000	18,000
R = 45q	8,000	12,000	16,000



Note that:

1. The break-even point, i.e. 320, divides the graph into a loss area where R < TC, and a profit area where R > TC.
2. The break-even sales units are on the x-axis. Break-even in sales value can be found by drawing a straight line from the break-even point to meet the y-axis, or 320 units x $45 = $14,400.

5.2.6 Target Net Profit

Continuing with the illustration, how many bags must PQR Ltd sell in order to make a net profit of \$6,000? What is the level of sales at this profit?

Using the equation:

$$\begin{aligned} R - TC &= \text{profit} \\ \$45q - \$20q - \$8{,}000 &= 6{,}000 \\ 25q &= 14{,}000 \\ q &= 14{,}000 \div 25 \\ q &= 560 \end{aligned}$$

560 bags must be sold in order to make a net profit of \$6,000.

The level of sales at this profit is

$$\begin{aligned} R &= 45 \text{ x } 560 \\ &= \$25{,}200 \end{aligned}$$

5.3 Exercise

1. A company makes a certain item. The selling price is \$15. The variable cost is \$5 per item. The fixed cost for the period is \$3000. Find the level of output and sales for (a) break-even (b) a profit of \$500.

2. A company has recorded the following data about the cost of production of one of its products.

   Fixed cost per month = \$1500
   Variable cost per unit = \$8
   Selling price per unit = \$16

   a. Find the level of sales per month when the profit is \$300.
   b. Find the level of sales per month at break-even.

3. A company uses two possible production processes, A and B. For A, the fixed cost per year is \$10,000 and the variable cost per unit \$2.50. For B, the corresponding costs are \$6,000 and \$5 respectively. If the selling price is \$4.50 irrespective of the process, find the break-even quantity for each process.

4. Find the break-even point graphically, for cost = $7 + 2x + x^2$ and revenue = $10x$.

   Use the range 0 to 10 for x at intervals of two units each.

5. Determine the break-even point graphically, when the supply is equal to the demand, where $S = 2q^2 - 40$ and $D = 25 - 3q$, q being the quantity for the range 0 to 10, for q at intervals of 2 units each.

6. A company manufactured a product whose profit P is given by

   $P = -q^2 + 160q - 1500$, q being the quantity produced.

   a. Draw the profit curve for q, for 0 to 160, at intervals of 20 units.

   b. Find the value of q for which the profit is zero.

   c. Find the value of q that maximises the profit.

7. For a particular item, the demand and supply are:

   Demand: $q = 1000 - 2p$

   Supply : $q = \frac{p^2}{400} - 200$

   a. Construct a table of values for each equation, for q and p, varying from 0 to 500 at intervals of 100 units each.

   b. Plot both on the same graph.

   c. Find the values of q and p for which supply is the same as the demand.

   d. Find the difference between the supply and demand when p = 450, showing this value on the graph.

Topic 6 – Presentation of Data

6.1 Introduction

6.1.1 Definition of Statistics

Statistics involves:

- collection;
- presentation;
- analysis;
- interpretation

of data as a basis for explanation, comparison and description.

6.1.2 Collection of Data

Data collected can be of two types.

- *Primary data* – information collected first hand for a specific purpose.
- *Secondary data* – information obtained from readily available sources.

6.1.3 Population and Samples

Population refers to every element in an observation that is of interest for data collection.

For example, if data on family size is needed, every citizen in for example, Singapore, is part of the population.

Due to time and cost constraints, obtaining information from every population member is usually replaced by sampling.

A sample refers to a certain number of elements that have been chosen from a population for observation.

For example, we choose to interview only 100 families instead of every family.

6.2 Presentation and Summary of Data

Data that has been collected must be well presented for analysis and interpretation. The objective is to highlight the chief features of the information.

6.2 Presentation of Data in Bar Charts

There are four different types of bar charts.

6.2.1 Simple Bar Chart

A simple bar chart is used to present only one variable.

The width of each bar and the gap between them are uniform.

Illustration:

Profit and losses of Tan Ltd. are as follows.

Year	Profit/Loss ($'000)
1990	1,000
1991	500
1992	– 1,000
1993	1,500
1994	2,500

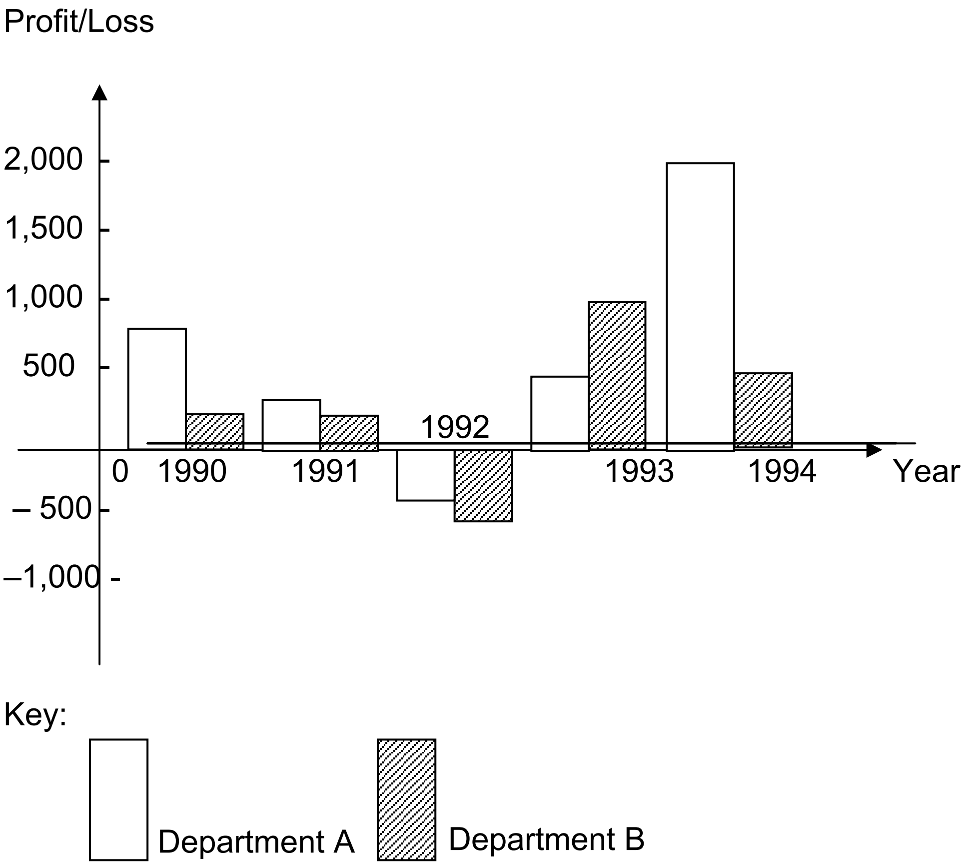
6.2.2 Multiple Bar Chart

A multiple bar chart is used to present two or more variables.

Illustration:

Net profit and losses ($'000) of Tan Ltd., divided between departments A and B, are as follows.

Year	Department A	Department B
1990	800	200
1991	300	200
1992	– 400	– 600
1993	500	1,000
1994	2,000	500



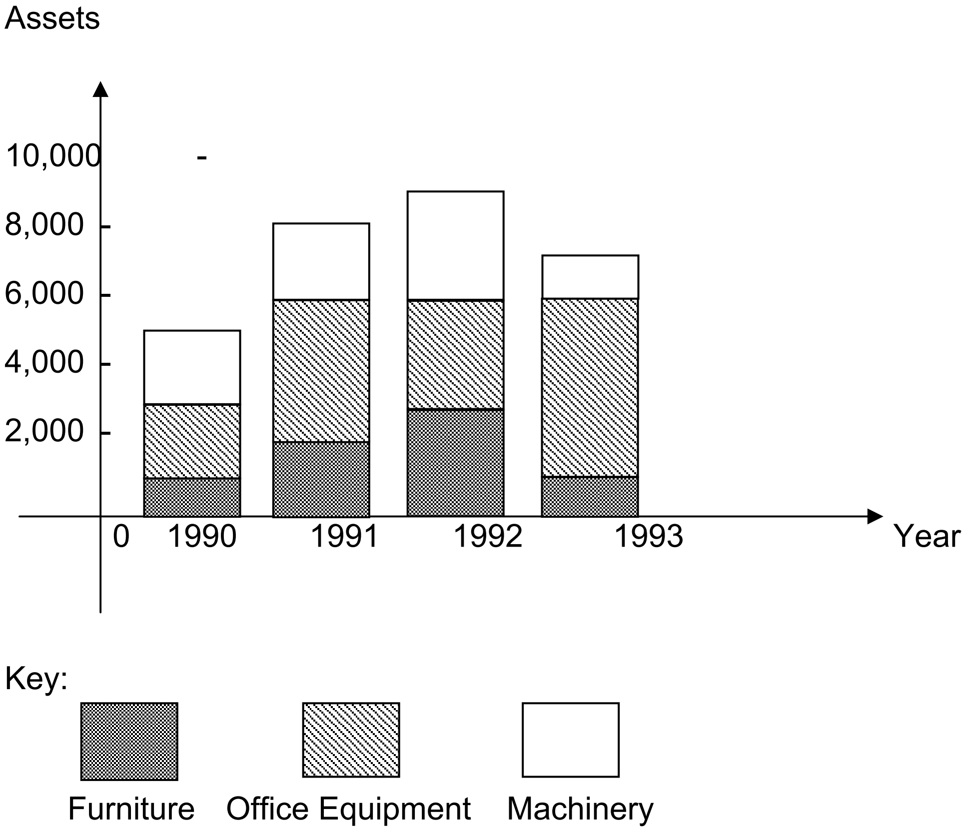
6.2.3 Component Bar Chart

This kind of bar chart illustrates the components that make up a total. Each component occupies a section of a bar in proportion to the total.

Illustration:

Total assets ($'000) of Acma Ltd. divided into different types, are as follows.

Year	Total Assets	Furniture	Office Equipment	Machinery
1990	5,000	1,000	2,000	2,000
1991	8,000	2,000	4,000	2,000
1992	9,000	3,000	3,000	3,000
1993	7,000	1,000	5,000	1,000



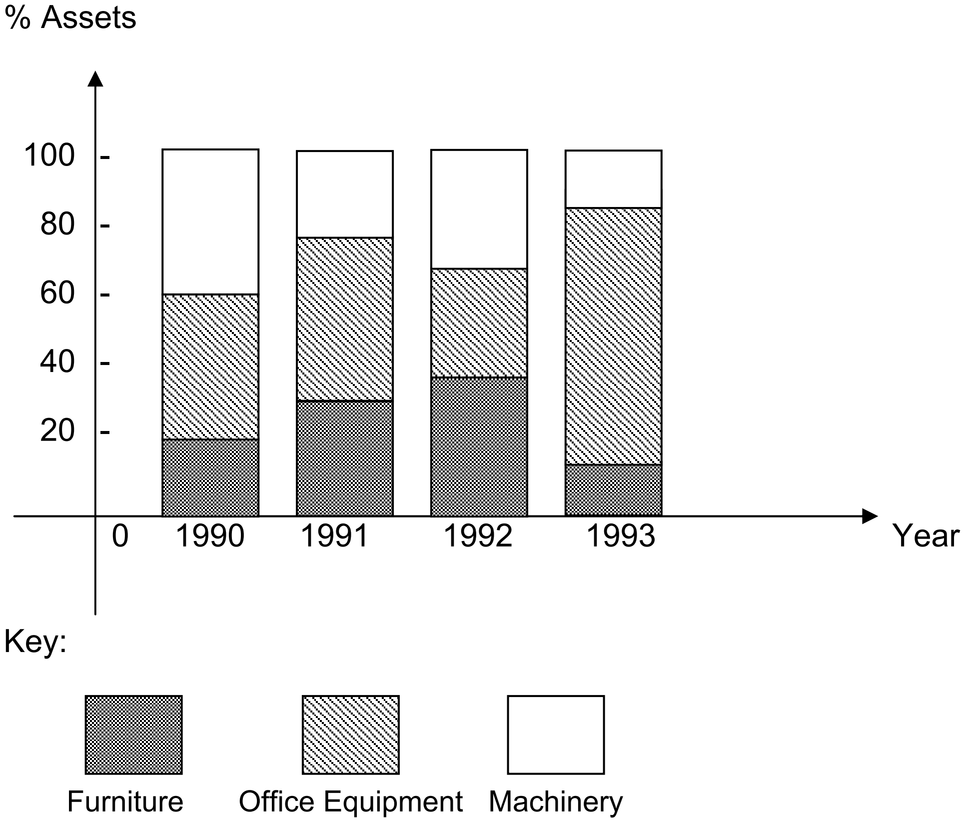
6.2.4 Percentage Component Bar Chart

Each component is expressed as a percentage in relation to the total.

Illustration:

Using the previous example:

Year	Total Assets	Furniture	Office Equipment	Machinery
1990	100%	20%	40%	40%
1991	100%	25%	50%	25%
1992	100%	33.3%	33.3%	33.3%
1993	100%	14%	72%	14%



6.3 Exercise

1. The table below shows the number of houses sold by a real estate agent in one year. Draw a simple bar chart representing this data.

Town	Yishun	Clementi	Jurong	Lakeside	Newton
No. issued	10	2	25	3	8

2. Draw a multiple bar chart representing the following data.

	Product Profit ($'000)		
Year	A	B	C
1980	1	2	2
1981	2	3	2
1982	3	4	5
1983	3	4	6

3. Draw (i) a component bar chart, (ii) a percentage component bar chart for the following data.

No. of Students			
Course	Term I	Term II	Term III
Accounting	50	40	35
Business Administration	100	90	80
Marketing	50	70	35

Comment on your graphs.

4. The following data gives the turnover of three different companies for three different years.

Turnover ($million)			
Year/Company	A	B	C
1990	15	20	30
1991	20	25	35
1992	25	30	40

Represent this data using (i) a component bar chart, (ii) a percentage component bar chart and (iii) a multiple bar chart.

6.4 Frequency Distribution Table

Data obtained from a survey or investigations is called raw data, before it is worked on. Some sets of raw data contain a limited number of data values, even though there may be many occurrences of each value. In this situation, the standard form into which the data is organised is known as a frequency distribution table.

A frequency distribution table consists of a list of data values, each showing the number of items having that value, i.e. the frequency.

The main aim of a frequency table is to summarise numeric data in a logical manner that enables an overall perspective of the data to be obtained quickly and easily.

Large collections of data can be grouped into classes, called class intervals.

The basic rules to follow in such classification are:

- Find the lowest and highest values of the variables
- Decide on the width of the class intervals
- Include all possible values of the variable

6.5 Ungrouped Frequency Table

Example:

The following data records the number of cars parked in 43 car parks in a city.

1	3	2	0	2	0	1	2	2	3
5	2	4	0	0	2	4	1	1	2
0	3	0	0	2	1	3	6	0	1
0	3	2	2	2	1	0	0	1	3
1	4	1							

Construct an ungrouped frequency table for this data by doing a tally count first.

Solution:

Smallest value = 0

Largest value = 6

Data value	Tally count	Frequency
0	~~IIII~~ ~~IIII~~ I	11
1	~~IIII~~ ~~IIII~~	10
2	~~IIII~~ ~~IIII~~ I	11
3	~~IIII~~ I	6
4	III	3
5	I	1
6	I	1
	Total	43

6.6 Stem and Leaf Diagram

This is an effective method of presenting individual data in classes. In this method, the original data is retained, in the sense that it can be identified from the diagram, unlike in a frequency table.

Illustration:

Consider the following heights, in cm, of 20 plants.

22	75	54	76	53	47	45	38	17	84
72	54	19	39	33	51	54	55	65	66

Draw a stem and leaf diagram for the above data.

Stem	Leaf
1	7,9
2	2
3	8,3,9
4	5,7
5	3,4,5,4,1,4
6	6,5
7	6,5,2
8	4

Key: 1│ 7 means 17.

1. The lowest data value is 17 and the highest data value is 84.
2. Choose classes of equal width, i.e., 10 - 19, 20 - 29, etc.
3. Let the stem represent tens.
4. Let the leaf represent units.
5. 22 can be represented as 2 in stem and 2 in leaf.
6. 75 can be represented as 7 in stem and 5 in leaf.
7. Complete the table in this way.

6.7 Exercise

Represent the following data using (i) an ungrouped frequency table, (ii) a stem and leaf diagram.

The weights in kg of 30 members of a health club are:

74	52	67	68	71	76	86	81	73	68
64	75	71	57	67	57	59	72	79	64
70	74	77	79	65	68	76	83	61	63

6.8 Grouped Frequency Table

The guidelines for constructing a grouped frequency table are:

- There should be about 10 class intervals.
- The width of each interval should be a ‘simple’ number (2,5,10,20 etc).
- The bottom score in each class interval should be a multiple of the width.
- All intervals should be the same width.

6.8.1 Classes

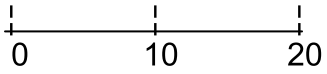
a. *Class interval* – this is a range of values defined by the lower class limit and upper class limit.

Example: 0 - 10

b. *Exclusive class interval* – class interval with no gap before the next class interval.

$$a < x < b$$

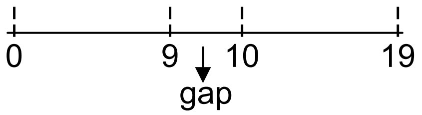
Example:
0 - <10
10 - <20
20 - <30



c. *Inclusive class interval* – class interval with gap before the next class interval.

$$a \le x \le b$$

Example:
0 - ≤ 9
10 - ≤ 19
20 - ≤ 29



Such groupings will leave a gap for data such as 9.2 which has no class interval to belong to. To remove the gap in order to draw a histogram, the inclusive class intervals are changed to class boundaries.

Class interval	Class boundary
0 - ≤9	–0.5 - 9.5
10 - ≤19	9.5 - 19.5
20 - ≤29	19.5 - 29.5

d. *Class mark or mid point* – this is the summation of the lower class limit and the upper class limit divided by two, for example, the class mark of 0 - 10 is:

$$(0 + 10) \div 2 = 5$$

e. *Number of classes – it* is usual practice to have between 6 and 15 classes. To obtain the number of classes, we divide the range by a predetermined class size.

f. *Class size* – it is usual practice to have class size in multiples of 5 or 10.

Number of classes = range ÷ class size

g. *Absolute frequency* – refers to the number of times the data falls within a class interval.

h. *Range:*

Range = largest value – smallest value

6.8.2 Data

a. *Discrete data* – data that can assume only exact values, e.g. number of cars on the road, houses, people.

b. *Continuous data* – data that can assume decimal places, e.g. weight, height, time.

Note: Inclusive or exclusive classes should be selected based on the following guidelines.

Inclusive classes	Exclusive classes
1. Discrete data, e.g. no. of bags sold in a sale. 2. Continuous and rounded off data, e.g heights of 50 plants to the nearest millimetre.	1. Continuous and not rounded off data, e.g. weights of students in kg.

Example:

The marks of 50 students were as follows.

58	47	48	70	55	30	25	21
41	10	8	66	69	5	38	37
51	43	43	15	19	27	29	30
60	15	40	48	10	77	79	88
99	81	87	16	73	83	31	44
98	100	84	20	99	48	26	41
40	53						

Classify the marks in suitable intervals by doing a tally count first.

Solution:

Marks are an example of continuous data.

Since it is not rounded off, we should use exclusive classes.

Range = largest value – smallest value
= 100 – 5 = 95

No. of classes = range ÷ class size
= 95 ÷ 10
= 9.5 (it is between 6 and 15)

Suitable class size is 10.

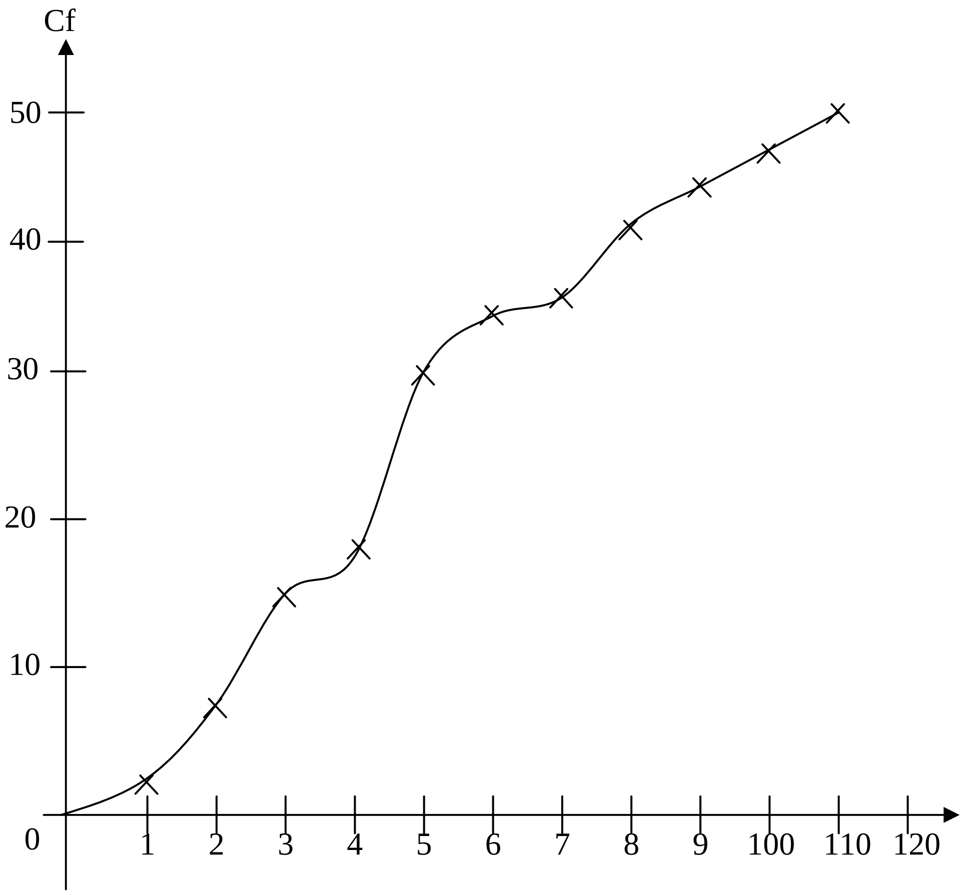
Frequency Table

Class interval	Tally count	Frequency
0 - 10	II	2
10 - 20	~~IIII~~ I	6
20 - 30	~~IIII~~ I	6
30 - 40	~~IIII~~	5
40 - 50	~~IIII~~ ~~IIII~~ I	11
50 - 60	IIII	4
60 - 70	III	3
70 - 80	IIII	4
80 - 90	~~IIII~~	5
90 - 100	III	3
100 - 110	I	1

6.9 Cumulative Frequency Curve (Ogive)

An ogive is a graphical depiction of cumulative frequency distribution.

Less than	Cumulative frequency
10	2
20	8
30	14
40	19
50	30
60	34
70	37
80	41
90	46
100	49
110	50



6.10 Exercise

1. A group of 50 employees is given a course in computer programming. The following data are the number of exercises completed by each one. Group these figures into a table with suitable classes.

13	9	5	11	14	6	5	8	11	13	10
16	15	3	19	18	9	9	5	12	13	12
15	9	18	12	16	7	12	13	11	18	15
9	21	9	11	6	12	12	10	16	2	14
10	17	8	15	11	12					

2. The data below is a record of the heights in cm of 85 men. Tabulate this data into suitable classes.

169	183	186	177	180	173	179	176	179	162	177
184	175	165	182	164	183	170	185	175	183	170
171	187	186	186	175	168	191	178	169	167	166
174	179	181	172	181	171	185	181	188	166	180
188	181	184	177	177	165	190	172	180	189	198
184	173	168	182	178	176	187	182	178	167	174
182	192	185	191	175	193	170	180	175	178	179
190	176	176	194	189	179	196	187			

3. The weights of 40 students are given in the table below. Construct a frequency table with suitable class intervals.

138	146	168	146	161	164	158	126	173	145	150
140	138	142	135	132	147	176	147	142	144	136
163	135	159	125	148	119	153	156	149	152	154
140	145	157	144	165	135	128				

4. The table below shows the distribution of marks gained by a group of 60 students.

Marks	0-10	10-20	20-30	30-35	35-40	40-45	45-50	50-65
Frequency	0	4	8	11	15	10	5	7

a. Draw a ‘less than’ cumulative frequency curve.

Grade ‘E’ was awarded to students scoring less than 33 marks, and grade ‘A’ was awarded to those scoring 55 or more. Estimate the % of students getting ‘A’ and ‘E’.

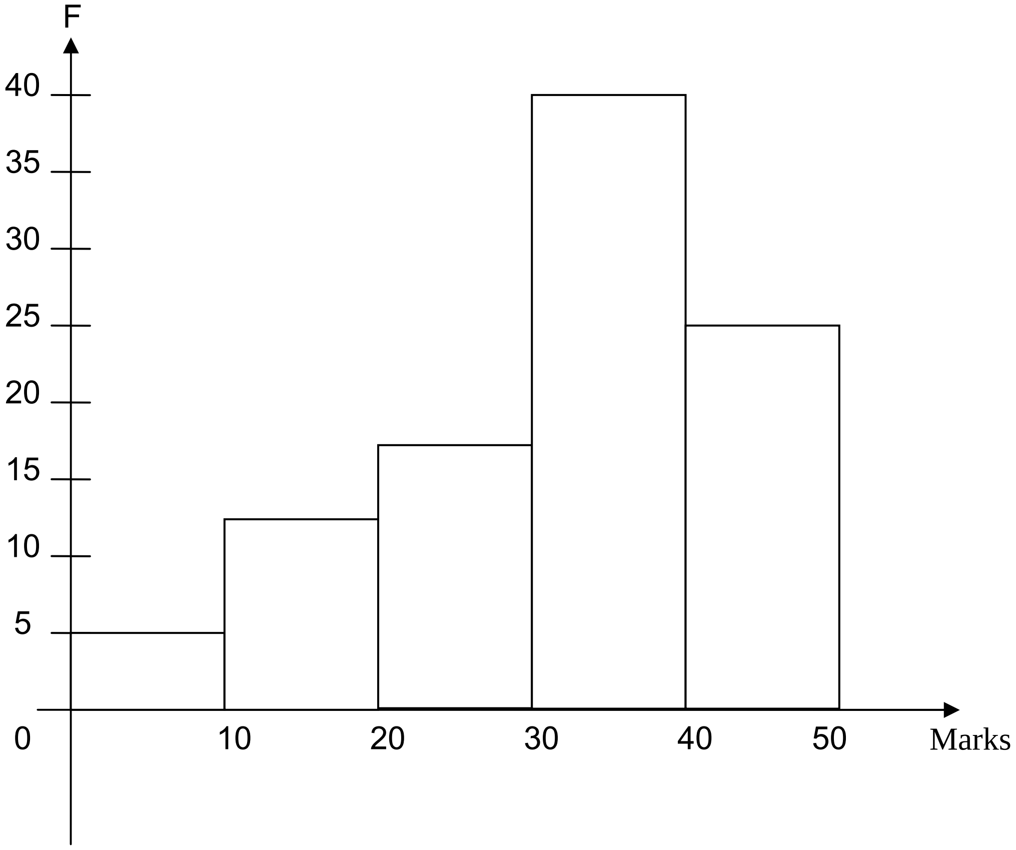
Topic 7 – Histograms and Pie Charts

7.1 Histograms

When grouped frequency distributions are drawn on a graph, we obtain a histogram. The absolute frequency is plotted on the Y-axis and the class interval is drawn on the X-axis. Care must be taken to ensure that there is no gap in between the class intervals. If there are gaps, then the class intervals are changed to class boundaries.

7.1.1 Example: Exclusive Classes

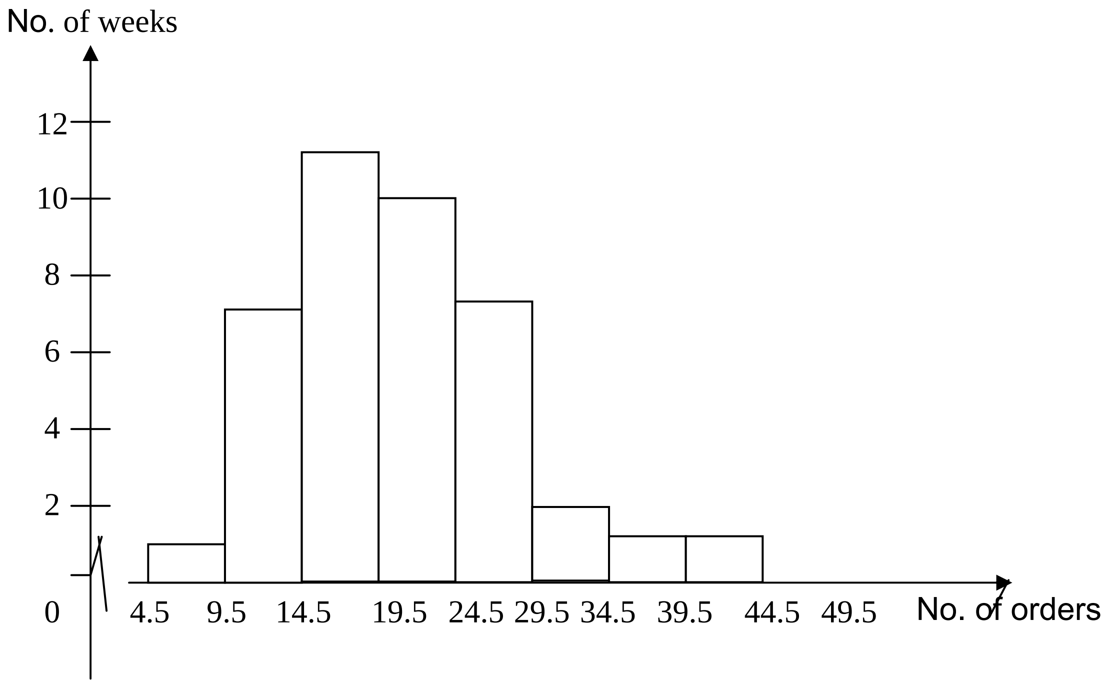
Marks	No. of students
0 - 10	5
10 - 20	12
20 - 30	18
30 - 40	40
40 - 50	25



7.1.2 Example: Inclusive Classes

No. of orders	5 - 9	10-14	15-19	20-24	25-29	30-34	35-39	40-44
No. of weeks	1	7	11	10	7	2	1	1

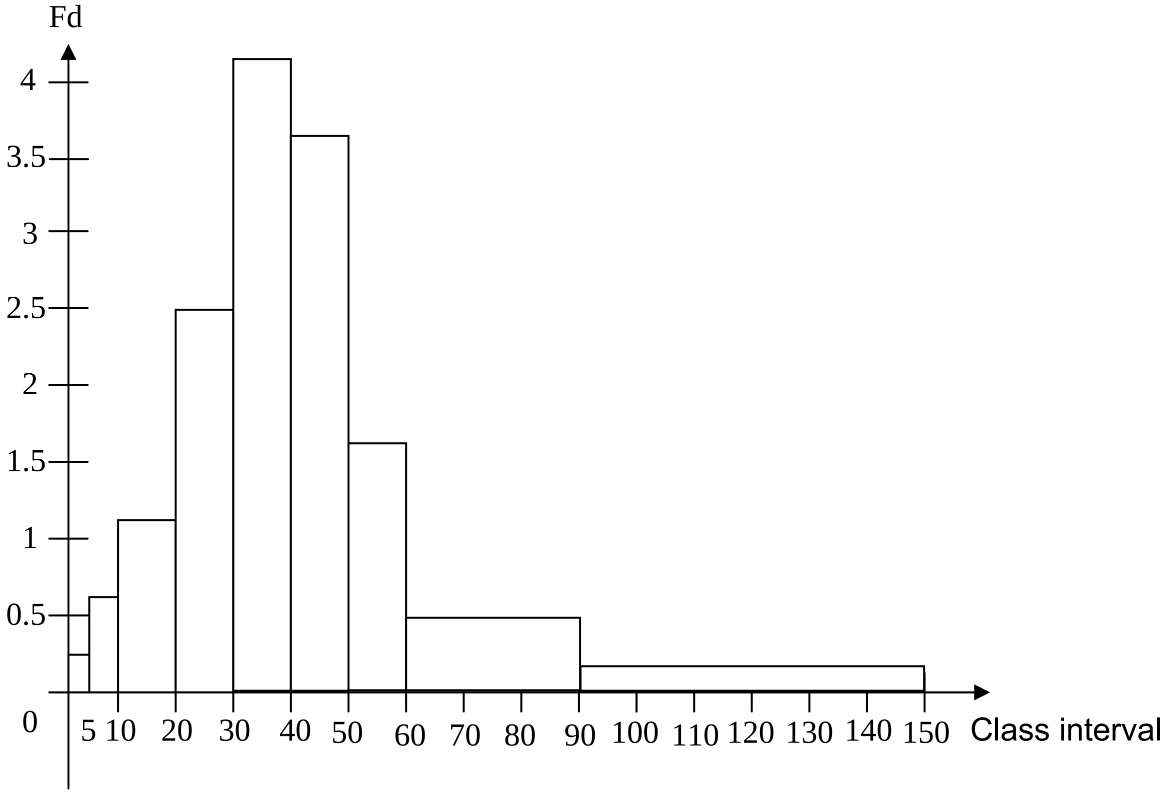
Class interval	Frequency	Class boundaries
5 - 9	1	4.5 - 9.5
10 - 14	7	9.5 - 14.5
15 - 19	11	14.5 - 19.5
20 - 24	10	19.5 - 24.5
25 - 29	7	24.5 - 29.5
30 - 34	2	29.5 - 34.5
35 - 39	1	34.5 - 39.5
40 - 44	1	39.5 - 44.5



7.1.3 Example: Exclusive and Unequal Classes

Class interval	0-5	5-10	10-20	20-30	30-40	40-50	50-60	60-90	90-150
Frequency	1	3	11	25	41	36	16	15	9

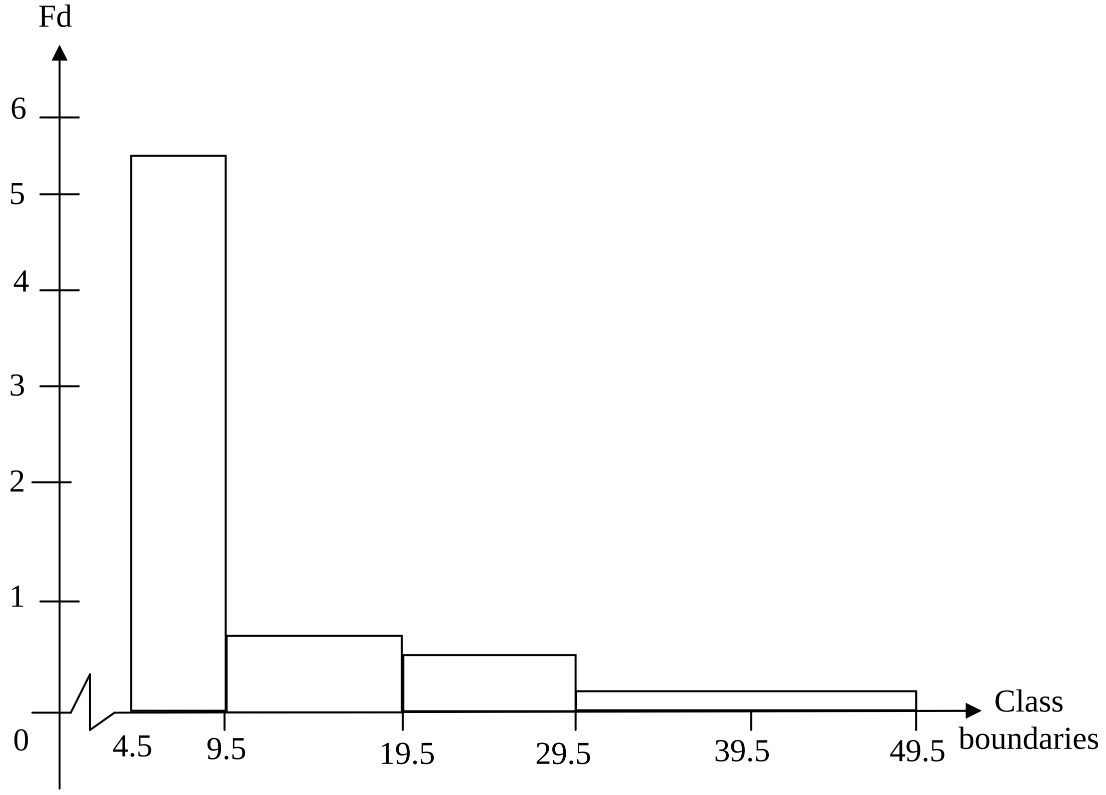
Class interval	Frequency	Class size	Frequency density
0 - 5	1	5	0.2
5 - 10	3	5	0.6
10 - 20	11	10	1.1
20 - 30	25	10	2.5
30 - 40	41	10	4.1
40 - 50	36	10	3.6
50 - 60	16	10	1.6
60 - 90	15	30	0.5
90 - 150	9	60	0.15



7.1.4 Example: Inclusive and Unequal Classes

No. of rejects	5 - 9	10 - 19	20 - 29	30 - 49
No. of days	28	9	7	2

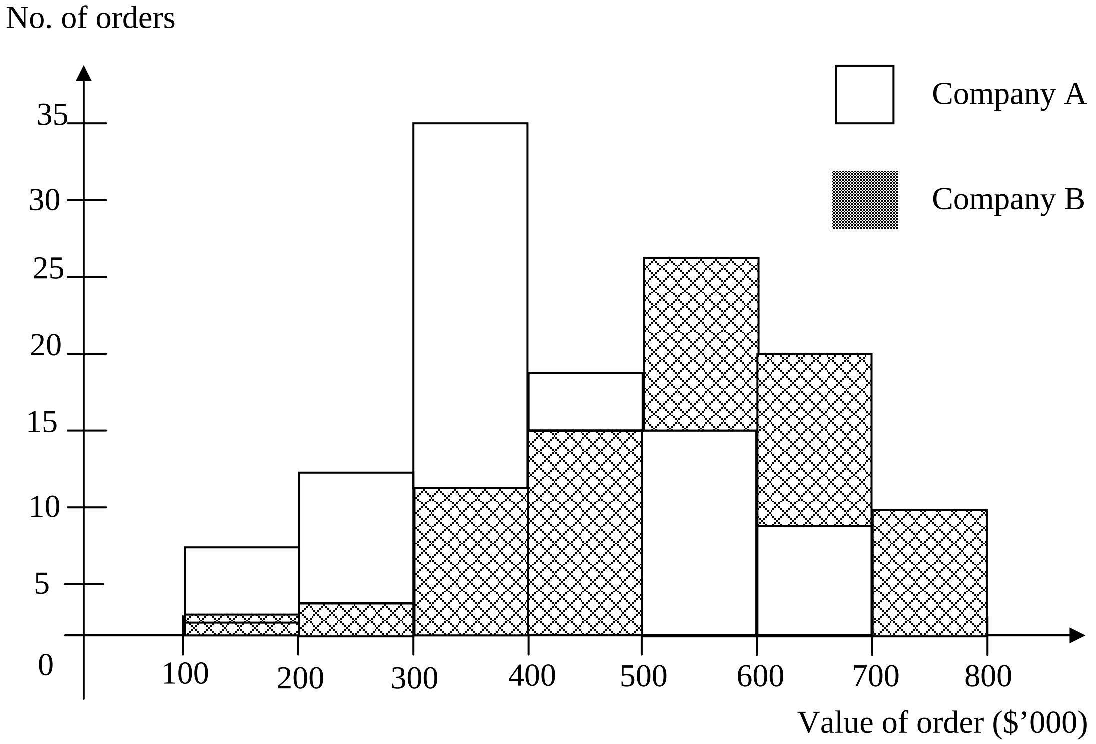
Class interval	Frequency	Class boundaries	Class size	Frequency density
5 - 9	28	4.5 - 9.5	5	28/5 = 5.6
10 - 19	9	9.5 - 19.5	10	9/10 = 0.9
20 - 29	7	19.5 - 29.5	10	7/10 = 0.7
30 - 49	2	29.5 - 49.5	20	2/20 = 0.1



7.1.5 Example: Superimposed Histogram

Represent the following data using a single histogram.

Value of orders ($'000)	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Company A	7	13	35	19	16	10	0
Company B	1	4	10	16	27	21	11



7.2 Pie Charts

A pie chart is a circular diagram which illustrates the proportional relationship between variables in a data set. Each segment is expressed in terms of a proportion of 360°. It illustrates the size of the variable relative to the total.

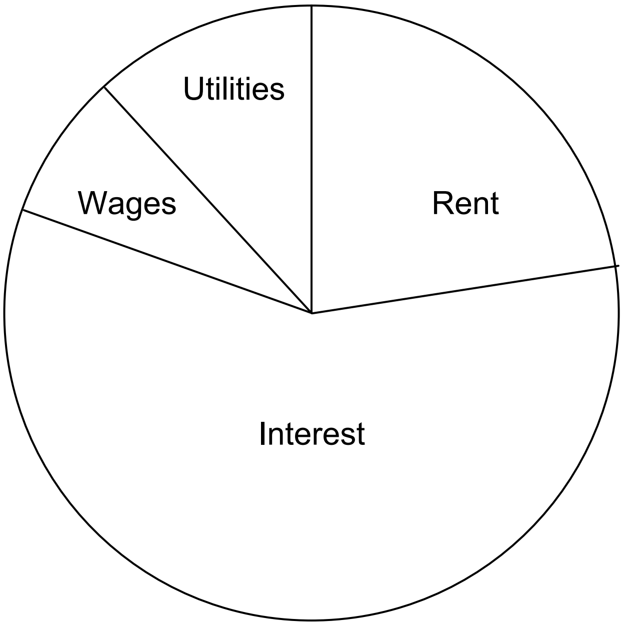
Example:

Data has been obtained from a profit and loss account regarding various expenses.

Expenses	Amount	Angle
Rent	\$200	$200/850 \times 360^o = 85^o$
Utilities	\$100	$100/850 \times 360^o = 42^o$
Wages	\$50	$50/850 \times 360^o = 21^o$
Interest	\$500	$500/850 \times 360^o = 212^o$
Total	\$850	360^o

Draw the corresponding pie chart in the space below.

Solution:



7.2.1 Example: Proportional Pie Charts

The table shows the expenditure (\$'000) of a company for two different years.

Year	Salary	Transport allowance	Advertising
1990	450	30	150
1995	600	50	200

If pie charts are drawn to illustrate and compare these data:

1. Calculate for each chart, the angle of the sector representing salary, transport allowance and advertising.
2. Given that for 1990, the radius of the pie chart is 5 cm, calculate the radius of the pie chart for 1995.
3. Draw both the pie charts.
4. Display the above data using another appropriate chart.

Solution:

1.

Year	Salary	Transport	Advertising	Total
1990	450 (257°)	30 (17°)	150 (86°)	(360°)
1995	600 (254°)	50 (21°)	200 (85°)	(360°)

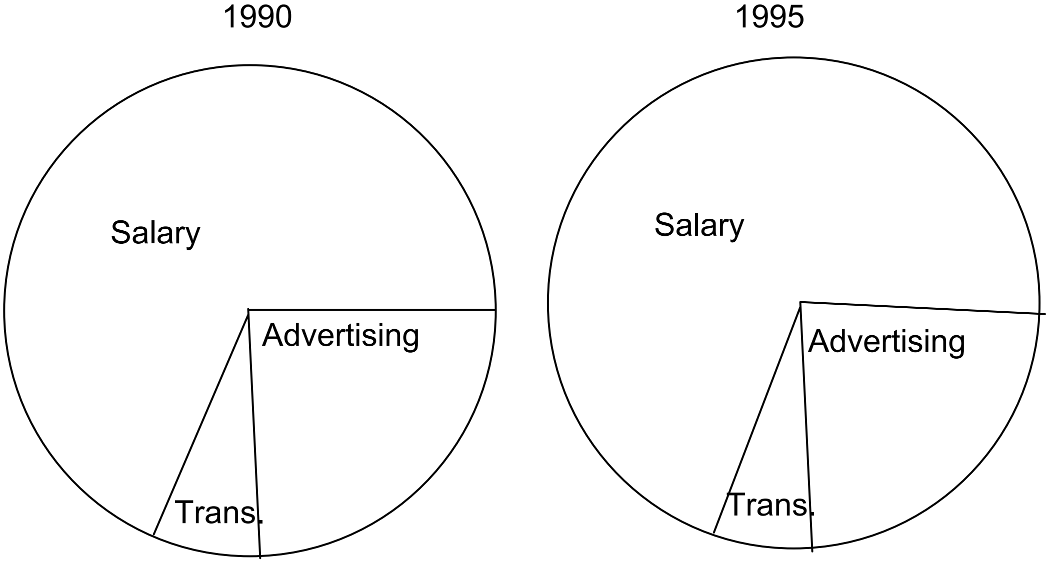
2. The radius is worked out using the formula:

$$\frac{R_1}{R_2} = \frac{\sqrt{T_1}}{\sqrt{T_2}}$$

Since R_1 = 5cm, T_1 = 630, T_2 = 850

R_2 = 5.8cm

3.



7.4 Exercise

1. Draw a histogram for each of the following.

a.

Profit	0-50	50-100	100-150	150-200	200-250	250-300
No. of shops	12	18	27	20	17	6

b.

Length of calls	10-12	12-14	14-16	16-18	18-20	(minutes)
No. of calls	40	109	54	22	15	

c.

Age	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
Dept A	5	7	7	18	23	14	10	6
Dept B	15	30	28	14	7	3	2	1

d.

Height of plants (cm)	0 - 0.4	0.4 - 0.8	0.2 - 1.2	1.2 - 2.0	2.0 - 2.4	2.4 - 2.8
No.	8	60	100	72	32	8

2. The table below gives the number of vehicles parked in five different car parks. Draw a pie chart showing the data.

Car park	A	B	C	D	E
No. of cars	21	38	9	12	4

3. The table below gives the percentages of sales of different toys in two toy stores.

Toy store	A %	B %
Dolls	19	21
Cars	36	27
Blocks	16	24
Guns	7	7
Soft toys	22	21
Total	2,910	20,810

   a. Construct a pie chart with a radius of 2.5 cm to represent the data for toy store A.

   b. Draw a proportional pie chart for toy store B.

Topic 8 – Measures of Central Tendency

8.1 Introduction

Measures of central tendency are measures of the location of the middle or the centre of a distribution, which may be representative of a whole group.

Examples:

- When trade unions are negotiating pay rises for their members, they often use the average wage as a standard on which to base increases.
- A business might use the average value of an order to forecast next year's turnover.
- An insurance company would use the average age at death to calculate the cost of a life insurance policy.

8.2 Arithmetic Mean

The mean is the average. It is the most useful of all the measures.

8.2.1 Formulae

1. *Individual data* – data with no frequency table.

$$\bar{x} = \frac{\sum x}{n}$$

2. *Ungrouped data* – data with frequency table without class intervals.

$$\bar{x} = \frac{\sum fx}{\sum f}$$

3. *Grouped data* – data with frequency table with class interval.

$$\bar{x} = \frac{\sum fm}{\sum f}$$ where m = mid point

8.2.2 Example: Individual Data

Data has been collected regarding the number of children in eight families in Singapore.

2, 2, 3, 4, 0, 6, 2, 1

$$\bar{x} = \frac{\sum x}{n}$$

$$\bar{x} = \frac{2+2+3+4+0+6+2+1}{8}$$

$$= \frac{20}{8} \quad = \quad 2.5 \text{ children}$$

8.2.3 Example: Ungrouped Data

Information regarding the number of children in 50 families in Singapore.

Number of children	Number of families
0	8
1	11
2	20
3	5
4	3
5	2
6	1

The mean is calculated as follows:

Number of children (x)	Number of families	Fx
0	8	0
1	11	11
2	20	40
3	5	15
4	3	12
5	2	10
6	1	6
	$\sum f = 50$	$\sum fx = 94$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{94}{50} = \text{1.88 children}$$

8.2.4 Example: Grouped Data

100 families were interviewed with regard to their income level and the following results were collected.

Income level $	Number of families
0 - 100	10
100 - 200	15
200 - 300	35
300 - 400	21
400 - 500	12
500 - 600	7

The mean is calculated as follows:

Income level	Frequency (f)	Mid point (m)	Fm
0 - 100	10	50	500
100 - 200	15	150	2,250
200 - 300	35	250	8,750
300 - 400	21	350	7,350
400 - 500	12	450	5,400
500 - 600	7	550	3,850
	∑f = 100		∑fm = 28,100

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{28{,}100}{100} = \$281$$

8.2.5 Features of Arithmetic Mean

- The value of the mean may not be the same as one of the original values.
- All values are taken into account in the calculation.
- Extreme values in the data affect the value of the mean.

8.3 Exercise

Calculate the mean for the following sets of data.

1. Individual observation

43, 75, 50, 51, 51, 47, 50, 47, 40, 48

63, 65, 67, 68, 69

5, 6, 6, 8, 8, 9, 11, 13, 14, 17

0.85, 0.88, 0.89, 0.93, 0.94, 0.96

1769, 1771, 1772, 1775, 1778, 1781, 1784

2. Ungrouped data

a.

x	1	2	3	4	5	6	7
f	4	5	8	10	17	5	1

b.

x	27	28	29	30	31	32
f	30	43	51	49	42	35

c.

x	121	122	123	124	125
f	14	25	32	23	6

3. Grouped data

a.

Class interval	5 - 9	10 - 14	15 - 19	20 - 24	25 - 29	30 - 34
f	4	6	12	10	7	1

b.

Class interval	101 - 104	105 - 108	109 - 112	113 - 116	117 - 120
f	13	18	21	12	6

8.4 Median

The median is the central value of the data when it is in ascending order. It is considered as an alternative to the mean.

8.4.1 Formula

1. *Individual data* – data with no frequency table.

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{ value}$$

2. *Ungrouped data* – data with frequency table without class intervals.

$$\text{Median} = \left[\frac{n+1}{2}\right]^{th} \text{ value where } n = \sum f$$

3. *Grouped data* – data with frequency table with class intervals.

$$\text{Median value} = \frac{n}{2}^{th} \text{ term}$$

$$\text{Median} = L_1 \left[\frac{n/2 - cfb}{f}\right] \times C$$

\+

where L_1 is the lower class limit of median class
n is the sample size
cfb is the cumulative frequency before the median class
C is the class size = $L_2 - L_1$
f is the frequency of the median class

8.4.2 Example: Individual Data

Data regarding the number of children in eight families in Singapore.

2, 2, 3, 4, 0, 6, 2, 1

Arrange in ascending order:

0, 1, 2, 2, 2, 3, 4, 6

The median item is calculated as follows:

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{ value}$$

$$= \left(\frac{8+1}{2}\right)^{th} \text{ value}$$

$$= \left[\frac{9}{2}\right]^{th} = 4.5^{th} \text{ value}$$

$$\text{The } 4.5^{th} \text{ value} = \frac{2+2}{2} = 2$$

Thus, the median = 2 children.

8.4.3 Example: Ungrouped Data

Information regarding the number of children in 50 families in Singapore.

Number of children	Number of families	Cumulative frequency	Position of terms	
0	8	8	1 - 8	
1	11	19	9 - 19	
2	20	39	20 - 39	← median
3	5	44	40 - 44	
4	3	47	45 - 47	
5	2	49	48 - 49	
6	1	50	50	
	$n = \sum f = 50$			

The median item is $\left(\frac{50+1}{2}\right)^{th}$ value = 25.5^{th} value = 2 children.

To locate the 25.5th item, we have to calculate the cumulative frequency. The 25.5th item belongs to families with two children. The median is therefore 2 children. Note that ∑f = n.

8.4.4 Example: Grouped Data

Exclusive Classes

	Income level $	No. of families	Cumulative frequency	P.O.T
	0 - 100	10	10	1 – 10
	100 - 200	15	25	11 – 25
Median →	200 - 300	35	60	26 – 60
	300 - 400	21	81	61 – 80
	400 - 500	12	93	81 – 93
	500 - 600	7	100	94 – 100
		n = ∑f = 100		

The median value is calculated as follows:

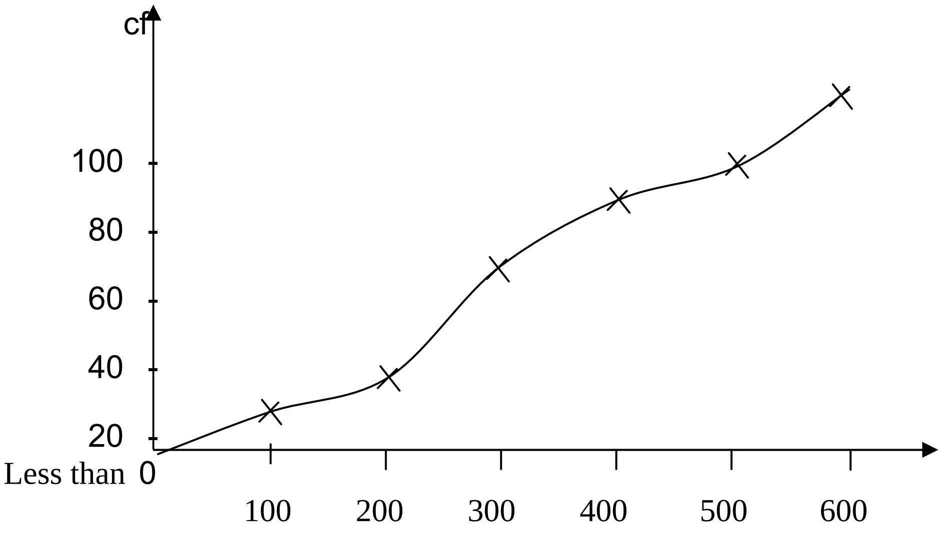
$$\frac{n}{2}^{\text{th}} \text{term} = \frac{100}{2}^{\text{th}} \text{term} = 50^{\text{th}} \text{ term}$$

and the median as follows:

$$L_1 + \left(\frac{\frac{n}{2} - \text{cfb}}{f} \quad \text{x} \quad C \right) \qquad = 200 + \left(\frac{\frac{100}{2} - 25}{35} \quad \text{x} \quad 100 \right)$$

$$= \$271.43$$

Graphical method – cumulative frequency curve:



Inclusive Classes

No. of calls made	f	Class-boundaries	cf	Position of terms
1 - 10	9	0.5 - 10.5	9	1 - 9
11 - 15	12	10.5 - 15.5	21	10 - 21
16 - 20	24	15.5 - 20.5	45	22 - 45 ← Median
21 - 25	16	20.5 - 25.5	61	46 - 61
26 - 40	14	25.5 - 40.5	75	62 - 75
total	n = 75			

Median value:

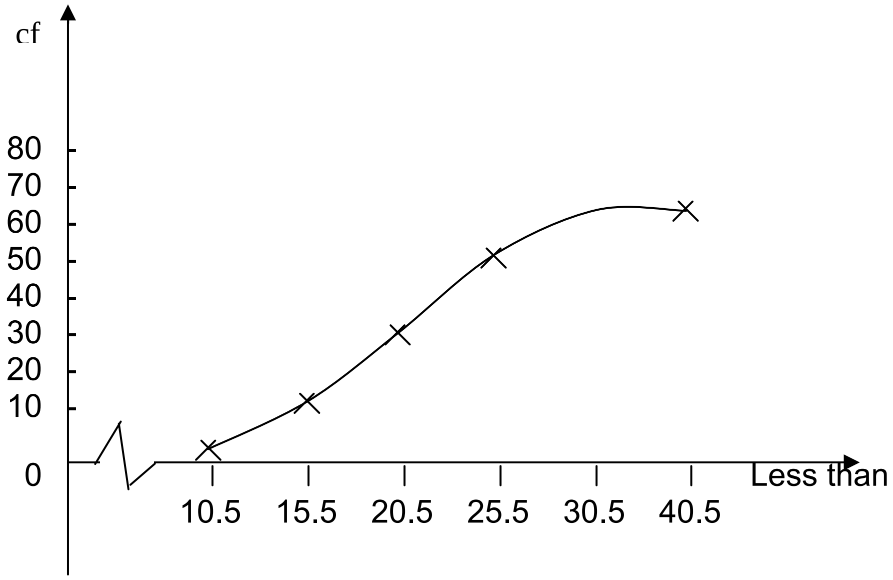
$$\frac{n}{2} = \frac{75}{2} = 37.5^{\text{th}} \text{ term}$$

Median:

$$L_1 + \left(\frac{\frac{n}{2} - \text{cfb}}{f} \text{ x } C \right) = 15.5 + \left(\frac{\frac{75}{2} - 21}{24} \text{ x } 5 \right)$$

$$= 18.94 \text{ calls}$$

Graphical method – cumulative frequency curve:



8.4.5 Features of Median

- It often expresses the average value more accurately than the mean, as it is not affected by extreme values in the data.
- Can be used when extreme values of a set are difficult, impossible or expensive to obtain.
- Can be used for non-numeric data.
- It often assumes a value equal to one of the original items.
- It is difficult to handle in more advanced statistical work.

8.5 Exercise

Calculate the median for the following:

1. Individual data

   7, 7, 2, 3, 4, 2, 7, 9, 31

   36, 41, 27, 32, 29, 38, 39, 43

   23, 26, 20, 27, 23, 28

   15, 13, 10, 19, 24, 12, 19, 26, 26

   3, 3, 5, 6, 8, 9, 12, 14, 19, 20, 24

2. Ungrouped data

   a.

x	0	1	2	3	4	5	6
f	15	24	18	12	8	2	1

   b.

x	1	2	3	4	5	6
f	12	9	8	13	9	9

3. Grouped data

   a.

$	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
f	3	7	10	9	5	2

   b.

cm.	3 - 5	6 - 8	9 - 11	12 - 14	15 - 17	18 - 20
cf	1	3	14	24	29	30

8.6 Mode

The mode is the value which occurs most often or has the largest frequency. It is useful in stock planning.

8.6.1 Formulae

1. *Individual data* – data with no frequency table.

   Mode = the value with the highest occurrence.

2. *Ungrouped data* – data with frequency table without class intervals.

   Mode = data with the highest frequency.

3. *Grouped data* – data with frequency table with class intervals.

$$\text{Mode} = L_1 + \frac{(fm - fb) \quad x \quad C}{(fm - fb) + (fm - fa)}$$

where L_1 is the lower class limit of modal class
fm is frequency of modal class
fb is frequency before the modal class
fa is frequency after the modal class
C $= L_2 - L_1$ = class size

8.6.2 Example: Individual Data

1. Mode exists and it is unique

   2, 3, 4, 2, 5, 2

   Mode = 2

2. Mode exists but it is not unique

   2, 3, 6, 4, 2, 3, 7

   Mode = 2, 3

3. Mode does not exist

   2, 4, 3, 6, 7, 1

   This data has no mode.

8.6.3 Example: Ungrouped Data

No. of children	No. of families
0	8
1	11
2	**20**
3	5
4	3
5	2
6	1

Mode = 2 children.

8.6.4 Example: Grouped Data

Exclusive classes:

Income level $	No. of families
0 - 100	10
100 - 200	15
200 - 300	**35** ← mode
300 - 400	21
400 - 500	12
500 - 600	7

Steps involved:

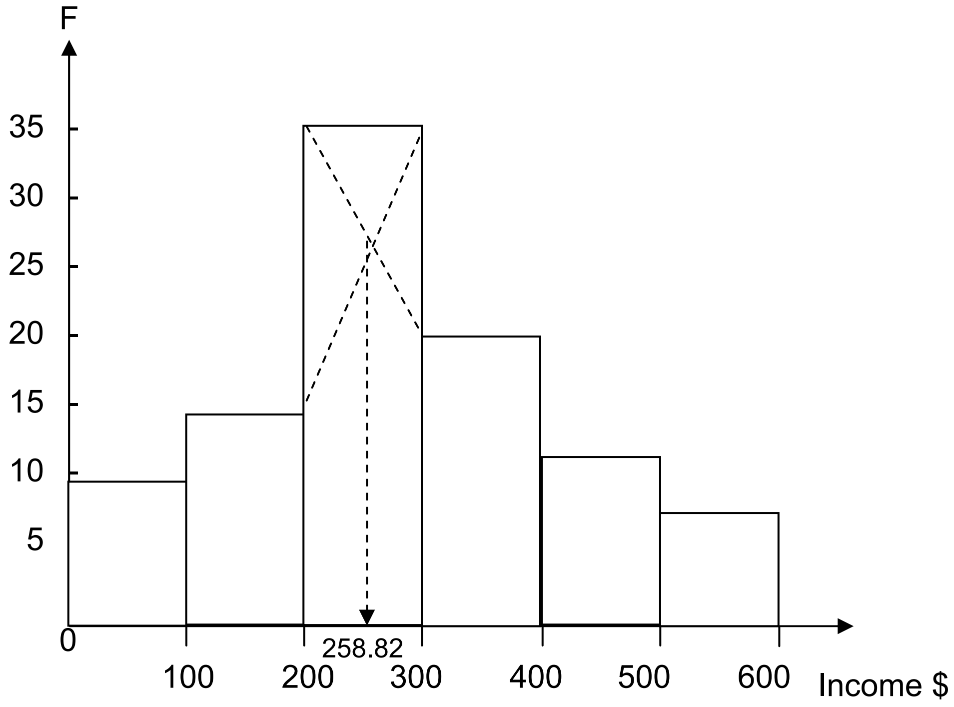
1. First, determine the modal class, i.e. the class with the highest frequency. In this case, the modal class is 200 - 300.
2. From the modal class:
   L_1 (lower class limit) = 200
   fm (frequency of modal class) = 35
   fb (frequency before modal class) = 15
   fa (frequency after modal class) = 21
   c (class size of modal class, i.e. 300 – 200 = 100)
3. Substituting into formula $L_1 + \frac{(\text{fm} - \text{fb}) \quad \text{x} \quad \text{C}}{(\text{fm} - \text{fb}) + (\text{fm} - \text{fa})}$

$$\text{Mode} = 200 + \frac{(35 - 15) \quad \text{x} \quad 100}{(35 - 15) + (35 - 21)}$$

$$= 200 + \frac{20 \times 100}{20 + 14}$$

$$= \$258.82$$

Graphical method – histogram:



8.6.5 Example: Unequal Classes

Class interval	0-5	5-10	10-20	20-30	30-40	40-50	50-60	60-90	90-150
Frequency	1	3	11	25	41	36	16	15	9

Class interval	Frequency	Class-size	Frequency density
0 - 5	1	5	1/5 = 0.2
5 - 10	3	5	3/5 = 0.6
10 - 20	11	10	11/10 = 1.1
20 - 30	25	10	25/10 = 2.5
^{L_1}30 - 40^{L_2}	41	10	41/10 = 4.1
40 - 50	36	10	36/10 = 3.6
50 - 60	16	10	16/10 = 1.6
60 - 90	15	30	15/30 = 0.5
90 - 150	9	60	9/60 = 0.15

← mode

$$\text{Mode} = L_1 + \frac{(fm - fb) \quad x \quad C}{(fm - fb) + (fm - fa)}$$

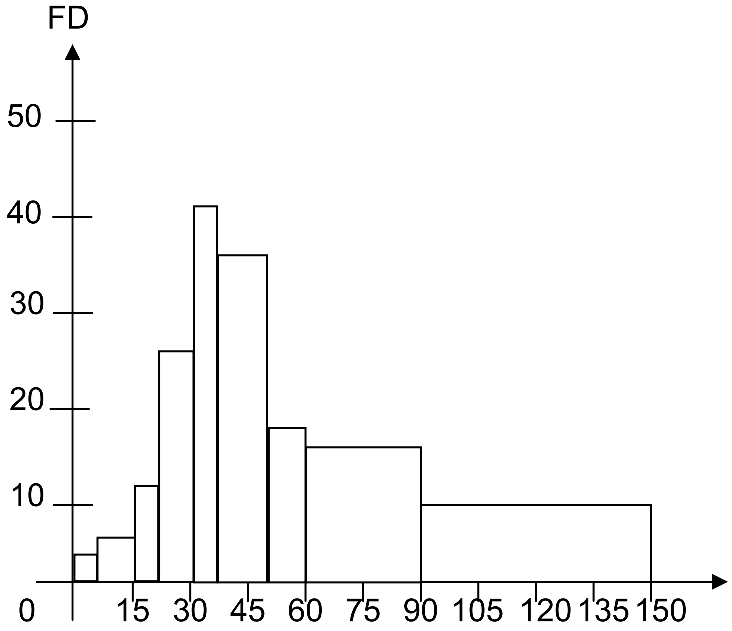
$$= 30 + \frac{(4.1 \quad - \quad 2\ .5)}{(4.1 \quad - \quad 3.6)} \text{ x } 10$$

$$= 30 + \frac{1.6(10)}{2.1}$$

$$= 30 + 7.619$$

$$= 37.62 \text{ (to 2 decimal places)}$$

Graphical method – histogram:



8.6.6 Example: Inclusive and Unequal Classes

No. of calls made	f	Class boundaries	C	FD = f ÷ c
1 -10	9	0.5 - 10.5	10	9/10 = 0.9
11 - 15	12	10.5 - 15.5	5	12/5 = 2.4
16 - 20	24	15.5 - 20.5	5	24/5 = 4.8 ← modal class
21 - 25	16	20.5 - 25.5	5	16/5 = 3.2
26 - 40	14	25.5 - 40.5	15	14/15 = 0.93
total	75			

$$\text{Mode} = L_1 + \frac{(fm - fb) \quad x \quad C}{(fm - fb) + (fm - fa)}$$

$$= 15.5 + \frac{(4.8 - 2.4)}{(4.8 - 2.4) + (4.8 - 3.2)} \times 5$$

$$= 18.5 \text{ calls}$$

8.6.7 Features of Mode

- It is useful when we need the most common value to represent the data.
- Easy to understand and calculate.
- It often assumes a value equal to one of the original items.
- It is difficult to handle in more advanced statistical work.

8.7 Exercise

Calculate the mode for the following.

1. Individual data
    a. 9, 1, 4, 8, 7, 8, 4, 2, 10, 9, 4
    b. 5, 8, 12, 10, 5, 3, 7, 5, 20, 10
    c. 20, 30, 25, 16, 5, 2, 2, 25
    d. 9.5, 9.7, 9.4, 9.9

Ungrouped data

a.

x	10	11	12	13	14	15	16
f	20	30	25	16	5	2	2

b.

x	0	1	2	3	4	5	6
f	10	5	7	9	34	41	5

2. Grouped data

a.

Age (years)	20-25	25-30	30-35	35-40	40-45	45-50
No. of employees	2	14	29	43	33	9

b.

Class interval	8-9	10-11	12-13	14-15	16-17
f	5	7	8	6	4

c.

Class interval	0-20	20-30	30-40	40-45	45-50	50-60
f	1	10	16	12	2	1

8.8 Compare and Contrast Mean, Median and Mode

1. The mean is an average indicator, which considers all the values in a set of data, and is the most useful statistical description. The average income of a nation gives an economic impression of its wealth, but when a distribution has extreme values, the mean is affected by such values. For example, a distribution of five persons’ weekly incomes:

   $500 $200 $250 $300 $6,000

   The mean income per week is $1,450.

   The mean is distorted or exaggerated by one person’s income of $6,000 (being the extreme amount). The majority are only earning between $200 and $500. Certainly the mean of $1,450 is misleading.

2. The median, being the middle item, implies that 50% of the data will be more than or equal to the median and 50% of the data will be less than or equal to the median. Therefore, it is a better indicator of average, when the mean is affected by extreme values. The tendency for the mean to be affected by extreme values does not make it a suitable basis for wage negotiation.

3. The mode is obtained by determining the data with the highest occurrence. Its use is mainly in stock planning. For example, the purchasing department in a shoe store will order more of the most modal shoe size.

Topic 9 – Measures of Dispersion

9.1 Measures of Dispersion

Averages alone are generally not enough to represent numeric data, since they only measure the central value, and do not consider the spread of the data. Data may vary from each other. So measures of dispersion should be used to measure the spread of the data.

9.2 Quartile Deviation

9.2.1 Definition

When a set of data is divided into four equal parts, each part is called a quartile.

- The first quartile of the data (25%) is called Q_1, the lower quartile.
- The second quartile of the data (50%) is called Q_2, or the median.
- The third quartile of the data (75%) is called Q_3, the upper quartile.

The interquartile range (IQR) is the difference between the third and first quartiles, and is a measure of dispersion.

9.2.2 Formulae

The formula for quartile deviation is: $\frac{Q_3 - Q_1}{2}$.

It measures the central 50% of the data.

1. Individual data – data with no frequency table.

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} \text{ value, } Q_3 = \frac{3(n+1)}{4}^{th} \text{ value, } QD = \frac{Q_3 - Q_1}{2}$$

2. Ungrouped data – data with frequency table without class intervals.

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} \text{ value, } Q_3 = \frac{3(n+1)}{4}^{th} \text{ value, } QD = \frac{Q_3 - Q_1}{2}$$

where $n = \sum f$

3. *Grouped data* – data with frequency table with class intervals.

Q$_1$ = $\frac{n}{4}^{\text{th}}$ value

$$Q_1 = L_1 + \left(\frac{\frac{n}{4} - \text{cfb}}{f} \text{ x } C \right) \text{ where } n = \sum f$$

Q$_3$ = $\frac{3n}{4}^{\text{th}}$ value

$$Q_3 = L_1 + \left(\frac{\frac{3n}{4} - \text{cfb}}{f} \text{ x } C \right)$$

$$QD = \frac{Q_3 - Q_1}{2}$$

9.2.3 Example: Individual Data

35, 40, 46, 51, 57, 66, 70, 75, 89

	Q$_1$ 25%	Median 50%	Q$_3$ 75%
To locate	$\frac{n+1}{4}$	$\frac{n+1}{2}$	$\frac{3(n+1)}{4}$

Q$_1$ is located using the formula:

$\frac{n+1}{4}$ i.e. $\frac{9+1}{4} = 2.5^{\text{th}}$ item

The average of the 2nd and 3rd data is $\frac{40+46}{2} = 43$

Q$_3$ is located using the formula:

$\frac{3(n+1)}{4}$ i.e. $\frac{3(9+1)}{4} = 7.5^{\text{th}}$ item

The average of the 7th and 8th data is $\frac{70+75}{2} = 72.5$

Quartile deviation is located using the formula:

$$\frac{Q_3 - Q_1}{2} \quad \text{i.e.} \quad \frac{72.5 - 43}{2} \quad = \quad 14.75$$

9.2.4 Example: Ungrouped Data

Information regarding the number of children in 50 families in Singapore.

	No. of children	No.of families(f)	Cumulative frequency (cf)	Position of terms
	0	8	8	1 - 8
Q1 →	1	11	19	9 - 19
Q3 →	2	20	39	20 - 39
	3	5	44	40 - 44
	4	3	47	45 - 47
	5	2	49	48 - 49
	6	1	50	
		$\sum f = 50$		

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} \text{ value}$$

$$= \left(\frac{50+1}{4}\right)^{th} \text{ value}$$

$$= 12.75^{th} \text{ value} \quad = \quad 1 \text{ child}$$

$$Q_3 = \frac{3(n+1)^{th}}{4} \text{ value}$$

$$= \frac{3(50+1)^{th}}{4} \text{ value}$$

$$= 38.25^{th} \text{ value} \quad = \quad 2 \text{ children}$$

$$\text{QD} = \frac{Q_3 - Q_1}{2}$$

$$= \frac{2-1}{2}$$

$$= 0.5$$

9.2.5 Example: Grouped Data

Exclusive classes:

Income level	No.of families(f)	Cumulative frequency (cf)	Position of terms	
0 - 100	10	8	1 - 10	
100 - 200	15	25	11 - 25	← Q1
200 - 300	35	60	26 - 60	
300 - 400	21	81	61 - 81	← Q3
400 - 500	12	93	82 - 95	
500 - 600	7	100	94 - 100	
	$\sum f = 100$			

$$\frac{n}{4} = \frac{100}{4} = 25 \quad \text{and} \quad \frac{3n}{4} = \frac{3(100)}{4} = 75$$

$$Q_1 = L_1 + \left(\frac{\frac{n}{4} - \text{cfb}}{f} \text{ x } C \right)$$

$$= 100 + \left(\frac{25-10}{15} \text{ x } 100 \right)$$

$$= \$200$$

$$Q_3 = L_1 + \left(\frac{\frac{3n}{4} - \text{cfb}}{f} \text{ x } C \right)$$

$$= 300 + \left(\frac{75-60}{21} \text{ x } 100 \right)$$

$$= \$371.43$$

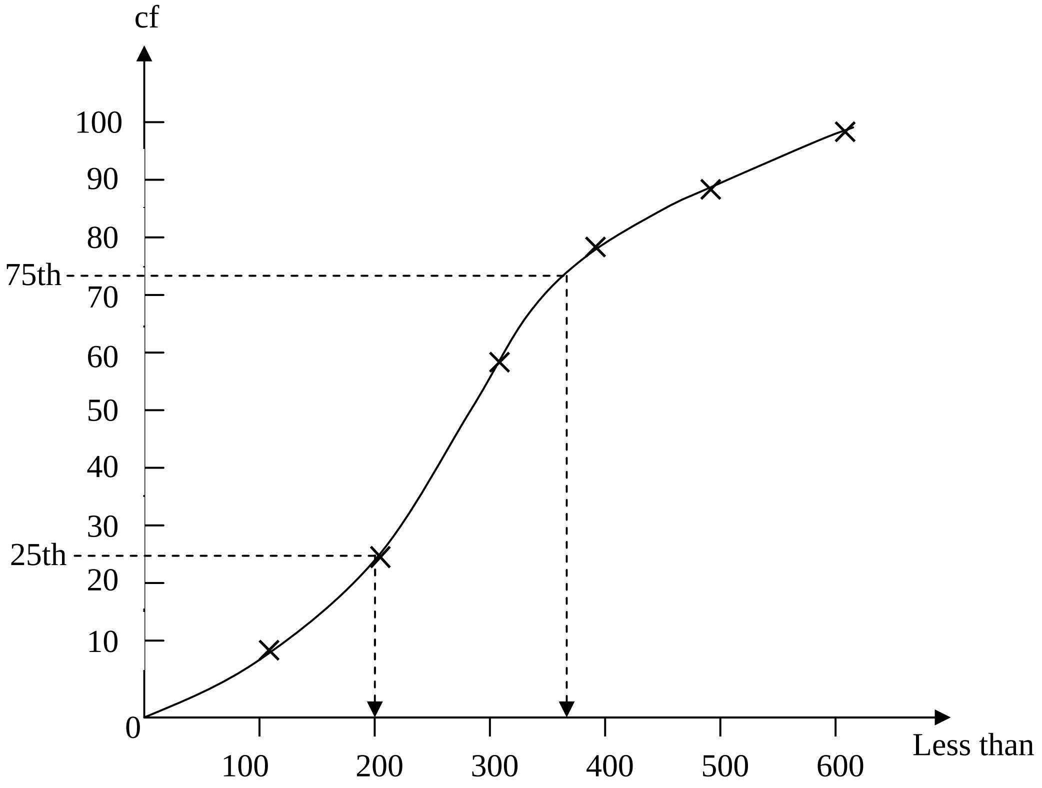
$$QD = \frac{Q3-Q1}{2}$$

$$= \frac{371.43-200}{2}$$

$$= \$85.72$$

Graphical method – cumulative frequency curve

Income level	No.of families(f)	cf	U.C.B
0 - 100	10	10	100
100 - 200	15	25	200
200 - 300	35	60	300
300 - 400	21	81	400
400 - 500	12	93	500
500 - 600	7	100	600
	$\sum f = 100$		



Inclusive classes:

No. of calls made	f	Class-boundaries	cf	Position of terms	
1 - 10	9	0.5 - 10.5	9	1 - 9	
11 - 15	12	10.5 - 15.5	21	10 - 21	← Q_1
16 - 20	24	15.5 - 20.5	45	22 - 45	
21 - 25	16	20.5 - 25.5	61	46 - 61	← Q_3
26 - 40	14	25.5 - 40.5	75	62 - 75	
Total	75				

$$\frac{n}{4} = \frac{75}{4} = 18.75 \qquad \text{and} \qquad \frac{3n}{4} = \frac{3(75)}{4} = \qquad 56.25$$

$$Q_1 \quad = \quad L_1 + \left(\frac{\frac{n}{4} - cfb}{f} \text{ x } C \right)$$

$$= \quad 10.5 + \left(\frac{18.75 - 9}{12} \text{ x } 5 \right)$$

$$= \quad 14.56 \text{ calls}$$

$$Q_3 \quad = \quad L_1 + \left(\frac{\frac{3n}{4} - cfb}{f} \text{ x } C \right)$$

$$= \quad 20.5 + \left(\frac{56.25 - 45}{16} \text{ x } 5 \right)$$

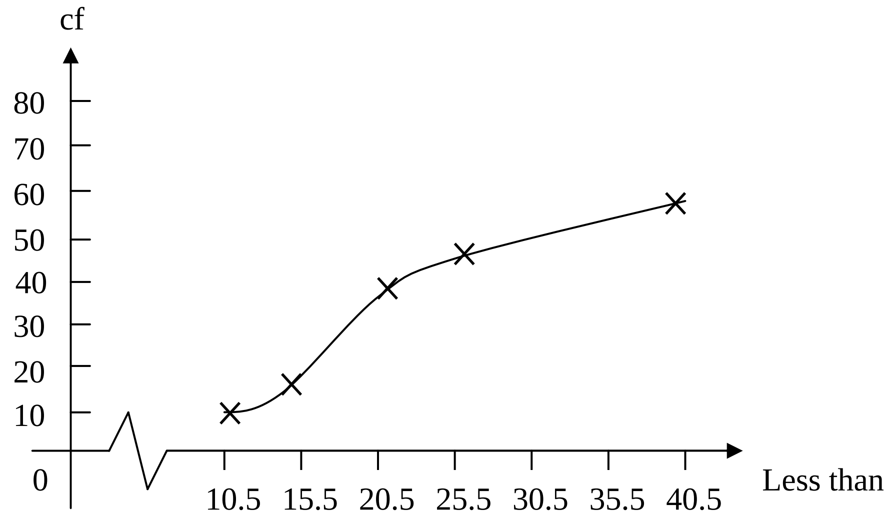
$$= \quad 24.02 \text{ calls}$$

$$QD \quad = \quad \frac{Q3 - Q1}{2}$$

$$= \quad \frac{24.02 - 14.56}{2}$$

$$= \quad 4.73 \text{ calls}$$

Graphical method – cumuative frequency curve:



9.3 Exercise

Calculate the quartile deviation for the following.

1. Individual data
   a. 43, 75, 48, 51, 51, 47, 50
   b. 32, 30. 24, 24, 36, 33, 29, 29, 30, 28, 30, 32, 32, 34, 28
2. Ungrouped data

a.

x	0	1	2	3	4
f	6	8	3	2	1

b.

x	0	1	2	3	4	5	6	7	8	9	10	11
f	4	8	11	12	21	15	10	4	2	2	1	1

3. Grouped data

Class interval	450 - 452	452 - 454	454 - 456	456 - 458
f	11	26	34	25

9.4 Standard Deviation

9.4.1 Definition

The standard deviation is the square root of the variance. It measures the spread of the data, as compared to the mean. When the standard deviation is small, it means that the data is concentrated near the mean. When it is large, the data is widely scattered from the mean.

9.4.2 Formulae

1. *Individual data* – data with no frequency table:

$$S = \sqrt{\frac{\sum (x-\overline{x})^2}{n}} \quad \text{OR} \quad \sqrt{\frac{\sum x^2 - (\overline{x})^2}{n}}$$

2. *Ungrouped data* – data with frequency table without class interval:

$$S = \sqrt{\frac{\sum fx^2}{\sum f} - (\overline{x})^2}$$

3. *Grouped data* – data with frequency table with class intervals:

$$S = \sqrt{\frac{\sum fm^2}{\sum f} - (\overline{x})^2} \quad \text{where m = mid point}$$

9.4.3 Example: Individual Data

Number of children in eight families in Singapore:

2, 2, 3, 4, 0, 6, 2, 1

Where x = $\frac{20}{8} = 2.5$, n = 8, $S = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$

$$S = \sqrt{\frac{(2-2.5)^2 + (2-2.5)^2 + (3-2.5)^2 + (4-2.5)^2 + (0-2.5)^2 + (6-2.5)^2 + (2-2.5)^2 + (1-2.5)^2}{8}}$$

$$= \sqrt{\frac{24}{8}}$$

$$= \sqrt{3} = 1.73$$

9.4.4 Example: Ungrouped Data

Information regarding the number of children in 50 families in Singapore:

No. of children	No. of families (f)
0	8
1	11
2	20
3	5
4	3
5	2
6	1

No. of children(x)	No. of families(f)	x^2	fx^2	Fx
0	8	0	0	0
1	11	1	11	11
2	20	4	80	40
3	5	9	45	15
4	3	16	48	12
5	2	25	50	10
6	1	36	36	6
	$\sum f = 50$		$\sum fx^2 = 270$	$\sum fx = 94$

$$\overline{x} = \frac{\sum fx}{\sum f} = \frac{94}{50} = 1.88 \text{ children}, \quad S = \sqrt{\frac{\sum fx^2}{\sum f} - (\overline{x})^2}$$

$$S = \sqrt{\frac{270}{50} - (1.88)^2}$$

$$= 1.36 \text{ children}$$

9.4.5 Example: Grouped Data

Income level	No. of families
0 - 100	10
100 - 200	15
200 - 300	35
300 - 400	21
400 - 500	12
500 - 600	7

Variance and standard deviation is calculated as follows:

Income level	No. of families(f)	Mid-point(m)	Fm	m^2	Fm^2
0 - 100	10	50	500	2,500	25,000
100 - 200	15	150	2,250	22,500	337,500
200 - 300	35	250	8,750	62,500	2,187,500
300 - 400	21	350	7,350	122,500	2,572,500
400 - 500	12	450	5,400	202,500	2,430,000
500 - 600	7	550	3,850	302,500	2,1171500
	$\sum f = 100$		$\sum fm =$ 28,100		$\sum fm^2 =$ 9,670,000

$$\overline{x} = \frac{\sum fm}{\sum f} = \frac{28100}{100}$$

= $281

Standard deviation S = $\sqrt{\frac{\sum fm^2}{\sum f} - (\overline{x})^2}$

$$S = \sqrt{\frac{9{,}670{,}000}{100} - (281)^2}$$

$$= \sqrt{17{,}739}$$

= $133.19

9.5 Coefficient of Variation

It is a useful statistic for comparing the degree of variation from one data series to another, when the mean values are significantly different. In these cases, it is better to use the coefficient of variation rather than standard deviation.

The coefficient of variation is :

$$(V\%) = \frac{S}{x} \text{ x } 100\%$$

Illustration:

	Company A	Company B
Sample size of products (n)	10	10
Average net profit (x)	$150	$120
Standard deviation (S)	$50	$45

Company A: V% $= \frac{\$50}{150} \text{x} \, 100$

$= 33.33\%$

Company B: V% $= \frac{45}{120} \text{ x } 100$

$= 37.5\%$

By examining standard deviation alone, it would seem that company A's products have greater variability in profit, but V% shows otherwise.

9.6 Exercise

Calculate the mean and standard deviation for the following.

1. Individual data
   a. 2, 4, 6, 8
   b. 6, 11, 14, 10, 8, 11, 9

2. Ungrouped data

a.

x	1	2	3	4	5	6	7	8
f	1	5	12	27	35	13	3	4

b.

x	1	2	3	4	5
F	11	10	5	3	1

3. Grouped data

a.

Class interval	0 – 4	5 - 9	10-14	15-19	20-24	25-29
f	1	14	23	21	15	6

b.

No. of books	31-35	36-40	41-45	46-50	51-55	56-60
No. of shelves	4	6	10	13	5	2

Topic 10 – Skewness and Symmetry

10.1 Introduction

Skewness is concerned with how symmetric or non-symmetric data is. To study the shape of a frequency distribution, the 'x' values are plotted on the x-axis and the frequencies on the y-axis. The shape of the graph can be symmetrical (no skew), positively skewed or negatively skewed. A positively skewed distribution has a long tail in the positive direction (to the right). A negatively skewed distribution has a long tail in the negative direction (to the left).This can be determined by using Pearson's coefficient of skewness, for which the formula is:

$$\text{sk} = \frac{3\,(\text{mean} - \text{median})}{\text{Standard deviation}}$$

10.2 Symmetrical or Normal Distribution

- This is a symmetrical bell shaped curve. (The Gaussian distribution).
- The mean, median and the mode are equal and lie in the centre of the curve.
- sk = 0.

Illustration:

Class interval	f	Mid point = m	fm	cf
0 - 10	2	5	10	2
10 - 20	8	15	120	10
20 - 30	30	25	750	40
30 - 40	8	35	280	48
40 - 50	2	45	90	50
			$\sum$ fm = 1250	$\sum$ f = 50

$$\bar{x} = \frac{\sum fm}{\sum f}$$

$$= \frac{1{,}250}{50} = 25$$

Next, find the median term:

median = n/2th value = 50/2th value = 25th value

and the median:

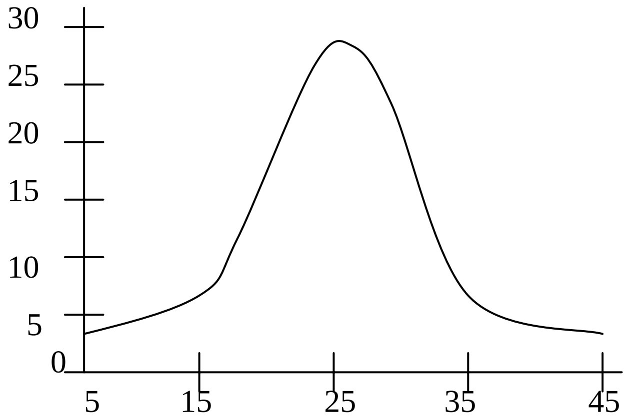
$$\text{Median} = L_1 + \left(\frac{\frac{n}{2} - \text{cfb}}{f} \times C \right)$$

$$= 20 + \left[\frac{\frac{50}{2} - 10}{30} \times (30 - 20) \right]$$

$$= 25$$

Hence: sk = 0

m	5	15	25	35	45
f	2	8	30	8	2



10.3 Positively Skewed Distribution

- The curve has a tail on the right side.
- The mean is affected by the extreme values and hence pulled towards the tail.
- Mode < median < mean.

Illustration:

Class interval	f	m	fm	fm^2	cf
0 - 10	2	5	10	50	2
10 - 20	8	15	120	1,800	10
20 - 30	30	25	750	18,750	40
30 - 40	5	35	175	6,125	45
40 - 50	3	45	135	6,075	48
50 - 60	2	55	110	6,050	50
	$\sum f = 50$		$\sum fm = 1,300$	$\sum fm^2 = 38,850$	

The mean:

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{1,300}{50} = 26$$

The median term:

Median = n/2th value = 50/2th value = 25^{th} value

and the median:

$$\text{Median} = L_1 + \left[\frac{\frac{n}{2} - cfb}{f} \times C \right]$$

$$= 20 + \left[\frac{\frac{50}{2} - 10}{30} \times (30 - 20) \right]$$

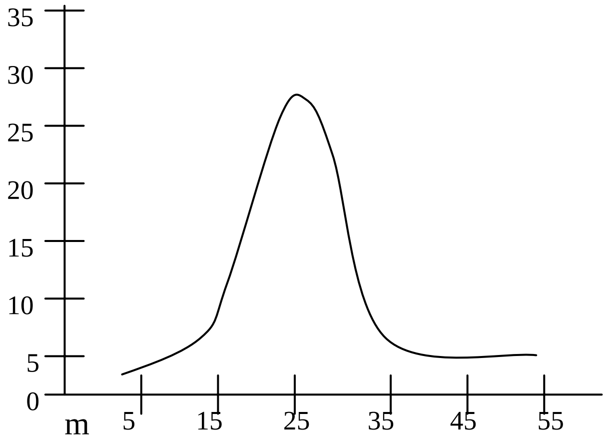
$$= 25$$

The standard deviation:

$$S = \sqrt{\frac{\sum fm^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{38{,}850}{50} - (26)^2} = 10.05$$

$$sk = \frac{3(26 - 25)}{10.05} = 0.2985$$

m	5	15	25	35	45	55
f	2	8	30	5	3	2



10.4 Negatively Skewed Distribution

- The curve has a tail on the left side.
- The mean is affected by the extreme values and hence pulled towards the tail.
- Mode > median > mean.

Illustration:

Class interval	f	m	fm	fm^2	cf
0 - 10	2	5	10	50	2
10 - 20	5	15	75	1,125	7
20 - 30	11	25	275	6,875	18
30 - 40	19	35	665	2,3275	37
40 - 50	25	45	1,125	50,625	62
50 - 60	40	55	2,200	121,000	102
60 - 70	18	65	1,170	76,050	120
	$\sum f = 120$		$\sum fm = 5,520$	$\sum fm^2 = 279,000$	

The mean:

$$\bar{x} = \frac{\sum \text{fm}}{\sum \text{f}}$$

$$= \frac{5,520}{120}$$

$$= 46$$

The median term:

Median = n/2th value = 120/2th value = 60th value

and the median:

$$\text{Median} = L_1 + \left[\frac{\frac{n}{2} - \text{cfb}}{\text{f}} \times C \right]$$

$$= 40 + \left[\frac{\frac{120}{2} - 37}{25} \right] \times (50 - 40)$$

$$= 49.2$$

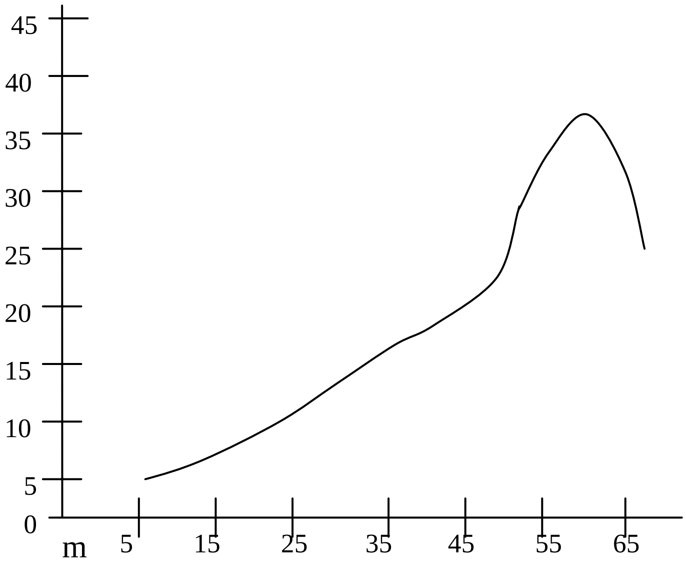
The standard deviation:

$$S = \sqrt{\frac{\sum \text{fm}^2}{\sum \text{f}} - (\bar{x})^2} = \sqrt{\frac{27,900}{120} - (46)^2}$$

$$= 14.46$$

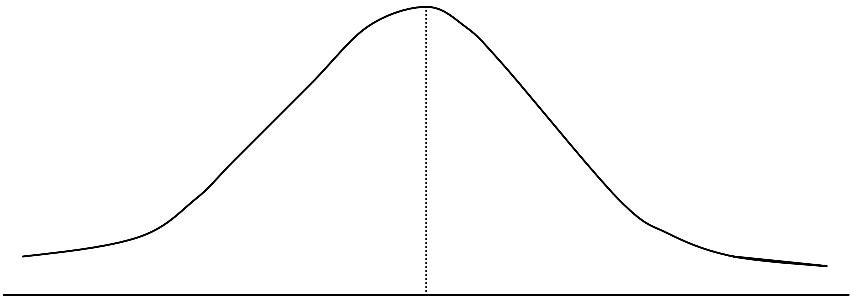
$$\text{sk} = \frac{3(46 - 49.2)}{14.46} = -0.664$$

M	5	15	25	35	45	55	65
F	2	5	11	19	25	40	18



10.5 Summary

1. Symmetical or normal distribution

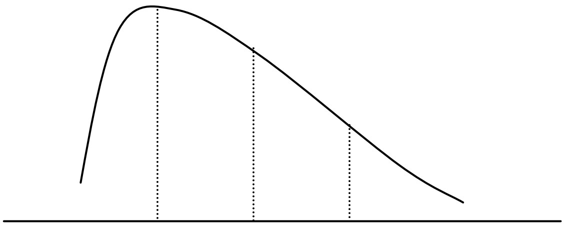


Mean = median = mode

Mean and standard deviation represent the data better.

sk = 0

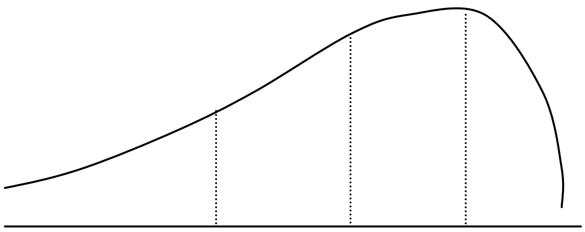
2. Positively skewed distribution



mode < median < mean

Median and quartile deviation represent the data better.

3. Negatively skewed distribution



mean < median < mode

Median and quartile deviation represent the data better.

Topic 11 Probability

11.1 Introduction to Probability

The world is full of uncertainties: there are in fact very few total certainties. Despite this we have to plan; we have to evaluate and allow for risks. We have to compare the reliability of systems and we have to attempt to forecast what is likely to happen.

Mathematicians have, over several centuries, developed and studied probability models. It is to these that we turn in order to represent an uncertain world and to satisfy the practical demands of living in it.

We need to have a grasp of both the principles and the practice of the application of probability models and theories for two reasons. Firstly our users, especially at management level, will need to make use of probability models for much of their planning and forecasting. Secondly, at both the practical and the theoretical levels, our own computer systems are 'complex states' and can only be evaluated by statistical methods and their behaviour forecast by probability models.

11.2 The Probability Scale

A principle of major concern is that the probability of any event occurring in the future *is evaluated on a scale from zero to one*.

The *totally impossible* event (although it is very difficult to make a list of examples since scientific progress has a habit of making yesterday's certainties obsolete), has the probability **0**.

Any *absolutely certain event*, again very difficult to list, has the probability **1**. Most events will have a probability, p, of $0 < p < 1$.

By whatever numerical process we evaluate past events and from them calculate what is likely to happen in the future, our *probability* answers cannot possibly be less than zero or more than one.

Probability is usually expressed on a scale of from 0….> 1. The extremes:

1 represents the probability of certainty;

0 represents the probability of impossibility.

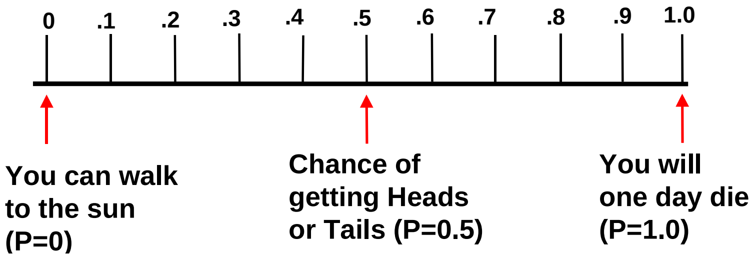


Figure 11.1 The Probability Scale

11.3 A Simple Model

At the simplest we may look at a single event and evaluate it. Suppose that our printer, according to the records of the past year, has a mean downtime of 20 minutes per eight-hour shift, what is the probability that we shall get through a full shift with a working printer?

First we must compare like with like, which means that the same set or the same measuring scale must be used. Minutes are not on the same scale as hours, so we must state the recorded mean as 20 minutes out of any 480 minutes.

The probability that the printer will break down is 20/480 and the probability that it will not is 460/480. We often use *p* to denote the probability that the event in which we are interested will take place and q to indicate the probability that it will not.

The first ‘law’ of probability is that:

$$p + q = 1$$

and nothing in any individual situation can change this. If we should ever arrive at any other result our conclusions are simply wrong. In the present example, if we first reduce our fractions to ‘lowest terms’ we see that a breakdown has:

$$p = 1 / 24 = 0.0416 = 4.2\%$$

whilst a shift unhindered by printer breakdown has:

$$q = 23 / 24 = 0.9583 = 95.8\%$$

and $p + q = 24/24$; 1.0 and 100% respectively.

The different ways of writing the fraction are all valid and equally commonly used so we should use the form which best suits the data.

Note: We must not confuse probability with *odds*. The odds against the printer breaking down, i.e. that it will not, are 23 to 1.

These statements of probability are mathematically accurate but they do not, of course, in any way influence the actual behaviour of the printer. Nor do they indicate what will happen on any particular shift; they refer only to the risk of breakdown. We deal not in predicting certainty but in forecasting what is likely to happen. The operations manager must either be able to evaluate the true risks, and take account of them in planning the workload, or just trust to luck that the system will be adequate.

A definition of probability is:

Probability P = $\frac{\text{The total number of favourable outcomes}}{\text{Total number of outcomes}}$

We can illustrate this in a Venn diagram (Figure 11.2), where the universal set represents the total outcomes, and subset illustrate the various outcomes.

- ε is the universal set.
- A and B are the subsets, known as events.

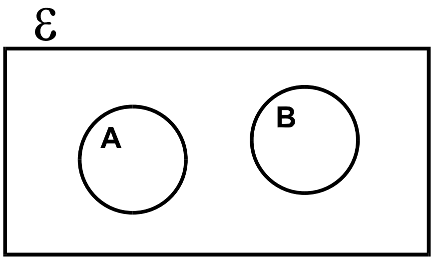


Figure 11.2 Venn Diagram – 1

Figure 11.3 shows events A and A' the complement of A.

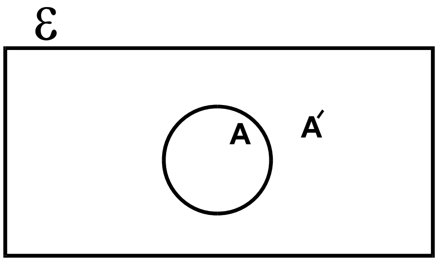


Figure 11.3 Venn Diagram – 2

Some events are *mutually exclusive*. They cannot happen simultaneously, for example, rolling a die, we can only get one of six results at a time.

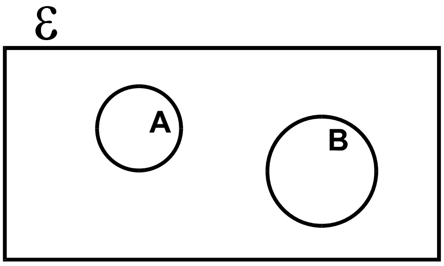


Figure 11.4 Mutually Exclusive Events

11.3.1 Addition Rule

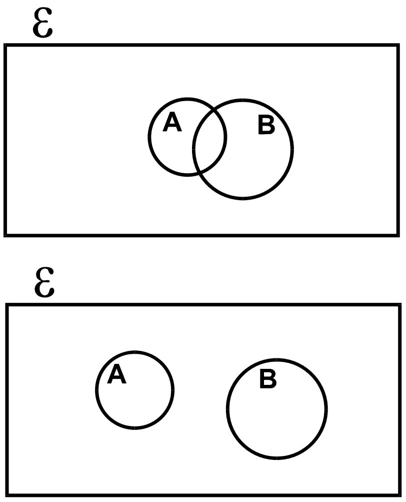


Figure 11.5 Addition Rule

If A and B are two events which occur in the same experiment, then we can say the probability of A or B or both happening can be written as:

P(A OR B), i.e.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When A and B cannot happen simultaneously, they are mutually exclusive and the formula becomes:

$$P(A \cup B) = P(A) + P(B)$$

Example:

The probability of drawing a diamond from a pack of cards is:

$$\frac{13}{52} \text{ i.e. } P(D) = \frac{13}{52}$$

Probability of drawing a king:

$$\frac{4}{52} \text{ i.e. } P(K) = \frac{4}{52}$$

Probability of drawing the king of diamonds:

$$\frac{1}{52} \text{ i.e. } P(Kd) = \frac{1}{52}$$

From the above, what would be the probability of drawing a diamond or a king?

We could get a simultaneous result to this, because we could draw the king of diamonds, so we can use the first formula:

$$P(D \text{ or } K) = P(D) + P(K) - P(D \cap K)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= \frac{16}{52}$$

$$= 0.31 \quad \text{i.e. } P(D \text{ or } K) = 0.31$$

If in an event:

P(A) is the probability that A happens;

P(A') is the probability that A does not happen,

then:

P(A happens or does not happen):

= P(A happens) + P(A does not happen)

$$1 = P(A) + P(A')$$

or:

$$1 - P(A) = P(A')$$

so the probability of A not happening = 1 - P(A)

Example:

The probability of drawing a club from a pack of cards is:

$$\frac{13}{52}$$

The probability of not drawing a club is:

$$\frac{39}{52}$$

From this:

$$\frac{13}{52} + \frac{39}{52} = 1$$

i.e.:

$$P(A) + P(B) = 1$$

You will come across this in many books, often written as:

$$P + Q = 1, \text{ or } P = 1 - Q \text{ or } Q = 1 - P$$

The above being understood, we will now take a look at a case where events A or B cannot happen simultaneously.

In this case our formula reduces to:

$$P(A \cup B) = P(A) + P(B)$$

Example:

Let a card be drawn from a set of 52 playing cards. What is the probability of drawing a red suit or a club:

P(R) = ½

P(C) = ¼

since they are disjoint sets.

Then $P(R \cup C) = P(R) + P(C) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Example:

What is the probability of drawing a heart or a spade from a pack?

$$P(H \cup S) = P(H) + P(S)$$

$$= \frac{13}{52} + \frac{13}{52} = 0.5$$

11.3.2 Multiplication

If A and B are two events, then the probability that both A and B occur:

$$P(A \cap B) = P(A) \times P(B)$$

For example, the probability of getting a 6 and a 2 from two throws of a die:

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Example

The probability of a king and an ace being selected from a pack of cards when the first choice has been replaced:

$$P(A \cap K) = \frac{4}{52} \times \frac{4}{52} = \frac{4}{676} = 0.0059$$

The probability of drawing one king and one ace from the pack of 52, when the king has not been replaced:

$$P(A \cap K) = \frac{4}{51} \times \frac{4}{52} = \frac{4}{663} = 0.006$$

11.3.3 Probability Spaces

In some problems it is an advantage to use the Venn diagrams.

Example

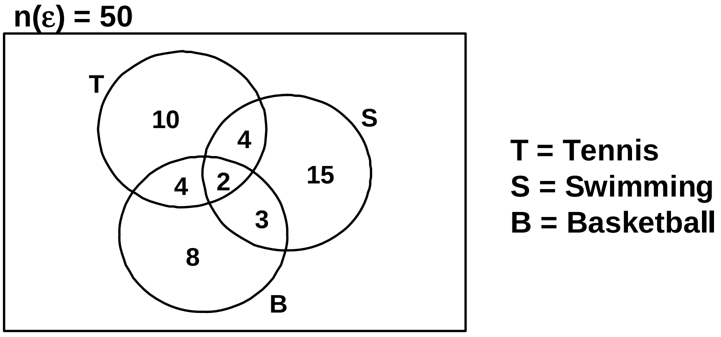


Figure 11.6 Probability Spaces

This diagram is made up of 50 students from College who gave their liking for various sports.

Find the probability, that when selected at random, a student indicate that he/she would:

(a) like tennis only;
(b) like basket ball and swimming;
(c) like all three.

(a) $P(x \in T) = \frac{10}{50} = 0.2$

(b) $P(x \in (B \cap S)) = \frac{5}{50} = 0.1$

(c) $P(x \in (T \cap B \cap S)) = \frac{2}{50} = 0.04$

Write down the probabilities of other components.

11.3.4 Relative Frequency

When we looked at the Histogram, we used them to examine frequencies within a given distribution, but we could not compare the categories with categories of another distribution. We could use percentages, but on the whole it is better to use relative frequencies.

Definition

Relative frequency histogram – to create a relative frequency histogram, the Relative Frequency (RF) for each category is calculated, i.e.:

$$\frac{\text{Actual frequency of the category}}{\text{Total frequencies}}$$

The relative frequency measures the way in which one particular category occupies the distribution as a whole and in what proportion.

If in one category (A) the frequency is 12, and the total frequencies for the distribution is 200, then

$$\text{RF of (A)} = \frac{12}{200} = 0.06$$

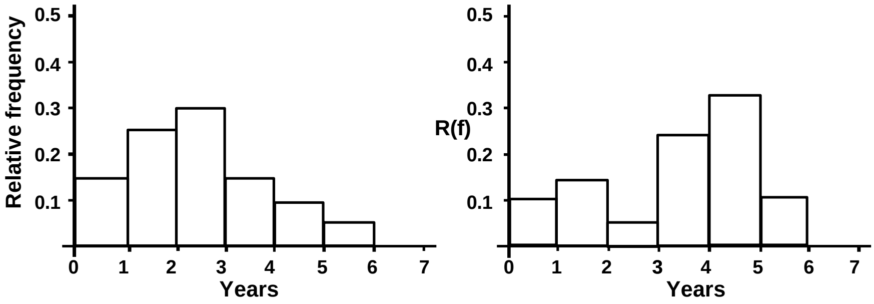
When these calculations have been done for all of the categories, then another histogram can be drawn, using the relative frequencies.

The total of all the relative frequencies for a given distribution will always be 1, no matter what the distribution is, or how many categories it may contain.

This means that individual parts of two relative frequency histograms are now capable of comparison.

It also means that the relative frequency histogram is a measure of the Probability for each category.

Figure 2.13 shows the ages of equipment in each of two workshops W6 and W8.



Age of W6 equipment

Age of W8 equipment

Figure 11.7 Relative Frequency Histogram

It can be seen at once that there is a far higher proportion of equipment aged between 4 and 5 years old in workshop W8 (relative frequency of 0.35, or 35%) than in workshop W6 (where the relative frequency is 0.1).

Had the actual number of machines of that age been relied upon then the respective sizes of the two workshops and the quantity of the equipment in each might have affected any conclusions reached.

The relative frequency measures the way in which one particular category occupies the distribution as a whole and in what proportion. We could easily ask the question:

> How likely is it, that a piece of equipment chosen at random from workshop W8, will be between 4 and 5 years old?

(*Note*: *Chosen at random* – all pieces of equipment were equally likely to have been chosen, i.e. no bias.)

The answer would be the proportion of workshop W8 equipment that is between 4 and 5 years old. Such likelihood is called a probability and in this case it has a value of 0.35.

By considering the relative frequency histogram and observing that it is really measuring the probability for each category, it becomes evident that the smallest value any probability can take is zero (as in the case of 6 to 7 year old equipment) up to a maximum value of 1 if all the equipment in the workshop were in the same age range.

11.4 Exercise

1. List the possible outcomes of a coin tossed three times. Calculate the probability of:

    (a) 2 heads and a tail;

    (b) 3 heads.

2. A card is drawn at random from a deck of 52 playing cards. What is the probability that it will be:

    (a) the king of spades;

    (b) a heart or a face card.

3. Three cards are drawn at random from a pack of cards. If after each draw they are not replaced, what is the probability of:

    (a) all three are spades;

    (b) all three are face cards.

4. The running times taken for a suite of ten batch programs were noted:

    2 programs took 1 minute to run;

    3 programs took 2 minutes to run;

    1 program took 3 minutes to run;

    4 programs took 4 minutesto run.

    On the basis of the this information, calculate the probability that another program in the same suite takes:

    (a) 3 minutes to run;

    (b) between 2 minutes and 4 minutes inclusive;

    (c) not longer than 2 minutes to run.

Topic 12 – Time Series

12.1 Time Series

When data is plotted over a long period of time, it is referred to as time series data.

Examples of this are annual sales, production, profit, turnover, etc.

Factors affecting time series data are:

- trend;
- seasonal factors;
- cyclical factors;
- random factors.

Inherent in the collection of data taken over time, is random variation. Factors such as festive seasons, weather conditions, recession etc will cause the data to fluctuate. Methods exist for reducing the effect of this variation.

12.2 Method of Moving Averages

Moving averages is used to smooth out short term fluctuations to highlight the longer term trends.

- Select a suitable period for moving average. The period to use depends on the natural cycle of the data, unless stated otherwise. Some examples are as follows.
  - Quarterly data – 4 moving average.
  - Weekly data – 5, 6 or 7 (depending on 5 or 6 or 7 week day) moving average.
  - Annual monthly data – 12 moving average.
- The period can either be odd (5 or 7) or even (4, 6, 12).

12.2.1 Odd Period

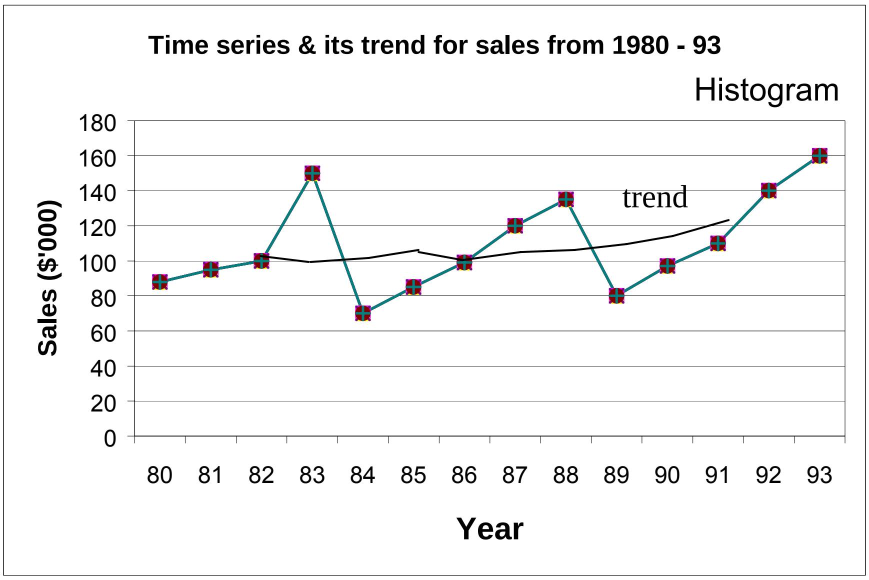
The following time series shows the sales of Mickey Mouse Ltd.

Year	$’000	Year	$’000	Year	$’000
1980	88	1985	85	1990	97
1981	95	1986	99	1991	110
1982	100	1987	120	1992	140
1983	150	1988	135	1993	160
1984	70	1989	80		

1. Calculate a 5-year moving average to isolate the trend.
2. Draw the time series and the trend on the same graph.
3. Comment on your graph.

Year	Sales ('000)	5-moving total	5-moving average
1980	88	-	-
1981	95	-	-
1982	100	503	100.6 (503÷5)
1983	150	500	100
1984	70	504	100.8
1985	85	524	104.8
1986	99	509	101.8
1987	120	519	103.8
1988	135	531	106.2
1989	80	542	108.4
1990	97	562	112.4
1991	110	587	117.4
1992	140	-	-
1993	160	-	-

Graph:



Comments – The original data fluctuates greatly. The trend shows an upward movement with substantial increase in year 89 to 90.

12.2.2 Even Period

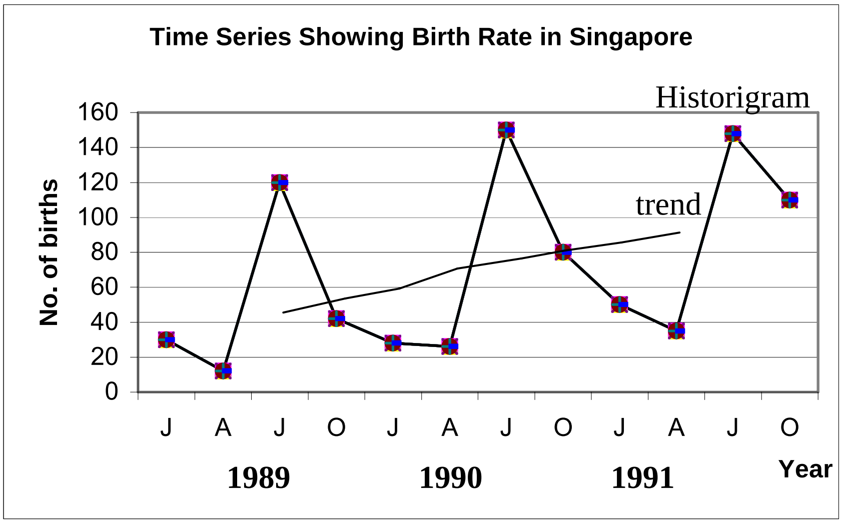
The data shows the birth rate, in thousands, in Singapore.

Year/Term	I	II	III	IV
1989	30	12	120	42
1990	28	26	150	80
1991	50	35	148	110

1. Calculate a suitable moving average to isolate the trend.
2. Draw the time series and the trend on the same graph.
3. Comment on your graph.

Year	Term	Original	4-moving total	Centred total	4-moving average
1989	I	30		-	-
	II	12		-	-
			204		
	III	120		203	50.75
			202		
	IV	42		209	52.25
			216		
1990	I	28		231	57.75
			246		
	II	26		265	66.25
			284		
	III	150		295	73.75
			306		
	IV	80		310	77.63
			315		
1991	I	50		314	78.5
			313		
	II	35		328	82
			343		
	III	148		-	-
	IV	110		-	-

Graph:



12.3 Exercise

1. The following data gives the total turnover for a company ($ million).

1970	5	1975	8	1980	20	1985	9
1971	8	1876	15	1981	16	1986	15
1972	6	1977	10	1982	15	1987	6
1973	12	1978	10	1983	6	1988	12
1974	4	1979	13	1984	18	1989	14

   a. For these data calculate the 5-year moving average.
   b. Plot the original data and the moving average on the same graph.
   c. Comment on what the graph shows.

2. The data below shows the sales ($'000) of an item.

Year/Term	Jan - Apr	May - Aug	Sep - Dec
1991	-	-	33
1992	67	12	39
1993	71	21	44
1994	78	27	47
1995	80	29	-

   a. By method of moving averages find the trend.
   b. Display the data on a graph and superimpose the trend.
   c. Comment on the movement of the original data and the trend line.

3. The following data gives the sales for the textile product of a company.

Sales ($‘000)				
Year/Quarter	I	II	III	IV
1988	144	141	179	192
1989	157	166	193	201
1990	167	151	195	189
1991	162	158	-	-

a. Draw a suitable graph for this time series.
b. Find the trend and add it to the graph.
c. Comment on what the graph shows.

4. The data below shows the sales ($’000) of a company.

Quarter	1982	1983	1984	1985	1986
1	-	71	73	76	84
2	89	90	91	97	106
3	106	108	111	122	-
4	78	79	81	89	-

a. Use the method of moving average to find the trend.
b. Draw the original data and the trend superimposed on a diagram. Hence comment on the sales of the company.